Complex Germen on invariant isotropic tori under the Hamiltonian phases flow with involution Hamilton functions

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1 Introduction

In several problems of the quantum and theoretical physics approximated solution to the partial differential equations which contain a small parameter in the higher derivative order are obtained, as well as to approximated eigenvalues and eigenvector of self-adjoint differential operator which depend on a small parameter. In such problems have been used the asymptotic methods [8, 7, 6], which nowadays are developed widely in several branches of the physics-mathematics. It is well known the success of asymptotic methods, e.g. with the quantification method was solved the older and sharp problem of the mechanic classic: calculation of the energetic level of the hydrogen atom [3].

In [7] over a 2n-dimensional phases space to obtain an asymptotic quasiclassical solution with respect to a small parameter on an isotropic tori k-dimensional ($k < n$) is obtained. This asymptotic on a torus is accomplished with a new geometric object which was called the Complex Germen .i.e. a family of complex planes with certain properties. Such object does not exist over any isotropic manifold. In such sense, V. P. Maslov put forward and solved the problem about its existence and construction techniques. At the same time, the uniqueness of Complex Germ has a great signification such that asymptotic be well defined. In [4] were completed the results obtained in [7], as well as have solved the uniqueness problem for the singular point and a closed trajectory of the Hamiltonian system. Further, was solved of existence and uniqueness of Germen on isotropic torus to Hamiltonian with cyclic variable.

Some more later, M. M. Nekhoroshev put forward this problem but on isotropic invariant torus with respect to Hamiltonian phases flows which come from k-functions in involution. This statement was partially solved in [9] establishing that if certain simplectic operator has a simple spectrum then the complex germ exist. In this work we solve this problem, providing a full solution, i.e. we present conditions for the existence and uniqueness of complex germ through the monodromy operator constructed in [9], but without the simple spectrum condition. We study also the Hamiltonian system with cyclic variables.

2 Preliminaries

Let M^{2n} be a $2n$-dimensional differentiable manifold of C^∞ with local coordinates (p,q). We assume that all the manifold are provided by C^∞ structures and the scalar functions and vectorial field as well. Let us introduce some basic result, details can be found in [1]. Let T_mM^{2n} be the tangent space to manifold M^{2n} at point m. Since $TM^{2n} = \cup_{m\in M^{2n}} T_mM^{2n}$ we can introduce natural manifold on it.

Definition 1. *It is called exterior form of degree two or 2-form on the manifold M^{2n} at point m to the application $\omega^2 : T_mM^{2n} \times T_mM^{2n} \to \Re$ bilinear and antisymmetric, i.e.*

- $\omega^2(\alpha x + \beta y, z) = \alpha\omega^2(x,z) + \beta\omega^2(y,z),$
- $\omega^2(x,y) = -\omega^2(y,x),$

for all $x, y, z \in T_mM^{2n}$ and $\alpha, \beta \in \Re$ (see [2]).

A 2-form ω^2 is closed if $d\omega^2 \equiv 0$, where $d : \Omega(M^{2n}) \to \Omega(M^{2n})$ is an operator of exterior differentiation on the space of the 2-form $\Omega(M^{2n})$. Besides, it is called non-degenerate if $\omega^2(x,y) = 0$ for all $x \in T_mM^{2n}$ then $y = 0$.

Definition 2. *A closed, non-degenerate and differential 2-form ω^2 on the manifold M^{2n} is called as a symplectic structure. The couple (M^{2n}, ω^2) is called symplectic manifold and the tangent space in each point m of the manifold is called as symplectic vectorial field whose symplectic structure is the restriction of ω^2 to $T_mM^{2n} \times T_mM^{2n}$.*

Example of symplectic structure is $M^{2n} = \Re^{2n}$ with $\omega^2 = dq \wedge dp$, where

$$dq \wedge dp(x,y) = \sum_{i=1}^{n} dq_i(x)dp_i(y) - dq_i(y)dp_i(x), \quad \forall x, y \in T_m\Re^{2n} \tag{1}$$

and for all $m \in \Re^{2n}$. In this case $T_m\Re^{2n}$ is identified with $\Re^{2n}$, where the 1-forms dq_i and dp_i are defined as

$$dq_i(x) = q_i(x) \quad \text{and} \quad dp_i(x) = p_i(x), \tag{2}$$

where q_i and p_i: $\Re^{2n} \to \Re$ are the coordinates system. i.e. $q_i(x)$ $(p_i(x))$ are i-th $(n+i$-th) coordinates of the vector x in a prefixed basis of the real linear space $\Re^{2n}$. The symplectic structure defined in this way is called standard.

Definition 3. *The coordinates of the local chart* $(q_1, \ldots, q_n, p_1, \ldots, p_n)$ *of a symplectic manifold are called canonical if the expression of the symplectic structure* ω^2 *in this coordinate system coincides with the standard.*

As consequence of the Darboux Theorem, in each point of a symplectic manifold there are a neighbour with canonical coordinates.

Definition 4. *Let* M^{2n} *be a symplectic manifold and let* $T_m M^{2n}$ *and* $T_m^* M^{2n}$ *be tangent and cotangent spaces at point* $m \in M^{2n}$*, we define the operator* $J : T_m^* M^{2n} \to T_m M^{2n}$ *as*

$$\omega^2(x, Ja) = a(x), \quad \textit{for} \quad x \in T_m M^{2n}; a \in T_m^* M^{2n}. \tag{3}$$

The operator J *defined in this way constitutes an isomorphism between vectorial spaces.*

Definition 5. *The Poisson bracket of two functions* F*,* G *over the manifold* M^{2n} *is defined as*

$$[F, G] = \omega^2(JdF, JdG), \tag{4}$$

where dF *and* dG *are the differentials 1-forms of* F *and* G *on* M^{2n}*.*

If $[F, G] = 0$, we say that functions F and G are in involution.
From Definition 4, we have

$$[F, G] = -dG(JdF) = dF(JdG). \tag{5}$$

Definition 6. *A Hamiltonian system is the triple* (M^{2n}, ω^2, H)*, where* (M^{2n}, ω^2) *is a symplectic manifold and functions* H *is defined on it. The field* JdH *is called the Hamiltonian vectorial field.*

The matrix of the Hamiltonian operator H in canonical coordinates is

$$\begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$$

where 0 and I_n denotes the zero and identity n-dimensional matrices. Thus, in canonical coordinates the Hamiltonian system takes the form

$$\dot{p} = H_q, \quad \dot{q} = -H_p, \tag{6}$$

where $H_q = (\frac{\partial H}{\partial q_1}, \ldots, \frac{\partial H}{\partial q_n})$ and $H_p = (\frac{\partial H}{\partial p_1}, \ldots, \frac{\partial H}{\partial p_n})$.

Let us assume that the solution of the Hamiltonian system (M^{2n}, ω^2, H) can be extended to $\Re$, i.e. for $-\infty < t < +\infty$. In this case, $g_H^t m$ denotes the value of the solution θ with initial condition $\theta(0) = m$, we obtain the application $g_H^t M^{2n} \to M^{2n}$ for t fixed. This application constitute a one parametric group of diffeomorphism, i.e. $g_H^0 m = m$ and $g_H^{t+s} m = g_H^t m \circ g_H^s m$, for all $m \in M^{2n}$.

This group is called flow of phases of the Hamiltonian system. Moreover, the application g_M^t is symplectic for each t fixed, i.e. $(g_H^t)^*\omega^2 = \omega^2$, where

$$(g_H^t)^*\omega^2(x, y) = \omega^2((g_H^t)_{*,m}x, (g_H^t)_{*,m}y), \tag{7}$$

for all $m \in M^{2n}$. Here $(g_H^t)_{*,m}$ denotes the derivatives of (g_H^t) for each $t \in \Re$ (see [2]).

Definition 7. *Let Λ a submanifold of M^{2n} and $T_m\Lambda$ the tangent space of Λ at point m. The submanifold Λ is called isotropic if the symplectic structure ω^2 is null on it, i.e., $\omega^2(x, y) = 0, \forall x, y \in T_m\Lambda$ and $m \in M^{2n}$.*

The submanifold is invariant respect to the Hamiltonian system (M^{2n}, ω^2, H) if $IdH(m) \in T_m\Lambda, \forall m \in \Lambda$, which in terms of the phases flow is rewritten as $g_H^t(\Lambda) = \Lambda$, for all $t \in \Re$, hence $(g_H^t)_{*,m}(T_m\Lambda) = T_m\Lambda$.

Let us consider the complexification of the linear space $\Re^2$ and a linear operator for a positive integer n. The complexification of $\Re^n$ is a n-dimensional linear space $(\Re^n)^{\mathbb{C}}$ constructed as follow: the point in $(\Re^n)^{\mathbb{C}}$ is denotes either by (x, y) or $x + iy$, where $x, y \in \Re^n$.

If $\alpha \cdot V$ denotes the multiplication of a scalar by $V \in \Re^n$ and $V + W$ the sum of two any vectors in $\Re^n$, the the multiplication of complex scalar $\alpha + i\beta$ by a vector $V + iW \in (\Re^n)^{\mathbb{C}}$ and the sum of two vectors $V_1 + iW_1$, $V_2 + iW_2$ are defined as:

$$(\alpha + i\beta) \cdot (V + iW) = (\alpha V - \beta W) + i(\alpha W + \beta V), \tag{8}$$

$$(V_1 + iW_1) \cdot (V_2 + iW_2) = (V_1V_2 - W_1W_2) + i(V_1W_2 + W_2V_2), \tag{9}$$

thus $(\Re^n)^{\mathbb{C}}$ constitutes an complex lineal space $((\Re^n)^{\mathbb{C}} = \mathbb{C}^n)$.

Remark. Similarly, for any real vectorial subspace P is possible to define its complexification $P^{\mathbb{C}}$.

It is well known, that between the tangent space $T_m(M^{2n})$ to submanifold 2n-dimensional M^{2n} and $\Re^{2n}$ there is an isomorphism $X : T_m(M^{2n}) \to \Re^{2n}$. Analogously, we can establish an isomorphism between the complexifications $T_m(M^{2n})^{\mathbb{C}}$ and $\Re^{\mathbb{C}}$.

Let us denote by $A : \Re^m \to \Re^d$ a $\Re$-linear operator. A complexification of the operator A is a $\mathbb{C}$-linear operator $A^{\mathbb{C}} : (\Re^m)^{\mathbb{C}} \to (\Re^d)^{\mathbb{C}}$ defined by the relation $A^{\mathbb{C}}(x+iy) = Ax+iAy$. The following relations are valid: $(A+B)^{\mathbb{C}} = A^{\mathbb{C}} + B^{\mathbb{C}}$, where A and B are $\Re$-linear operators.

Let us introduce the following concepts: let ω^2 be a real simplectic structure on the manifold M^{2n}. We call complexification of ω^2 defined on $T_m(M^{2n})^{\mathbb{C}}$, for all $m \in M^{2n}$ to the form $\omega^{\mathbb{C}} : T_m(M^{2n})^{\mathbb{C}} \times T_m(M^{2n})^{\mathbb{C}} \to \mathbb{C}$, given by

$$(\omega^2)^{\mathbb{C}}(u + iv, x + iy) = (\omega^2(u, x) - \omega^2(v, y)) + i(\omega^2(u, y) + \omega^2(v, x)), \tag{10}$$

$\forall u + iv, x + iy \in T_m(M^{2n})^{\mathbb{C}}$.

Remark 8. *Since ω^2 is antisimetric then $(1/2i)(\omega^2)^{\mathbb{C}}(x, \bar{x})$ is real for all $x \neq 0$ in $T_m(M^{2n})$. We denote $(\omega^2)^{\mathbb{C}}(x, y) = [x, y]$. It is possible to verify that the complexification of the standard simplectic structure on $T_m(M^{2n})$ is the standard simplectic structure on $T_m(M^{2n})^{\mathbb{C}}$ at any point m of the manifold M^{2n} ([5]).*

3 Statement of the problem

Let M^{2n} a $2n$-dimensional simplectic manifold and $F_1, \dots, F_k$ a family of function defined on it: $F_j : M^{2n} \to \Re$, $j = 1, \dots, k$ such that $k < n$ which stay involution on M^{2n}. Let the k-Hamiltonian system (M^{2n}, ω^2, F_j). Let $g^t_{F_j}, t \in \Re; j = 1, \dots, k$ the Hamiltonian flux phases of such system. Since the function $F_j, j = 1, \dots, k$ are involution then the flux of phases commute i.e. $g^t_{F_i} \o g^t_{F_j} = g^t_{F_j} \o g^t_{F_i}$, $\forall t \in (-\infty, +\infty)$ and $\forall i, j = 1, \dots, k$.

We assume that

- The Hamiltonian flux phases $g^t_{F_j}$, $j = 1, \dots, k$ are global, i.e. it are defined for all $t \in (-\infty, +\infty)$. It is valid for example assuming that the manifold M^{2n} be compact.
- The tori $\Lambda^k = S^1 \times, \dots, \times S^1$ (where S^1 denotes the unit circle), $k < n$ is a k-dimensitional isotropic submanifold of M^{2n} ($\Lambda^k \subset M^{2n}$) e invariant respect to the Hamiltonian system $(M^{2n}, \omega^2, F_j), j = 1, \dots, k$.
- The differential $dF_1, \dots, dF_k$ are lineal independent on each point of the manifold M^{2n}.

For the construction of asymptotic solution of several partial differential equation in [7] the concept of Complex Germ on isotropic manifold is introduced. The issues of existence and uniqueness of such object is treated in this work. The main difficulty is that not always exist the Germ over any isotropic manifold.

Definition 9. *A Complex Germ over the isotropic tori $\Lambda^k (k < n)$ is a smooth map on Λ^k, $r^n : m \to r^n(m), \forall m \in \Lambda^k$, such that to each point $m \in \Lambda^k$ correspond a n-dimensional complex subspace $r^r(m)$ of the complexification of the tangent space to M^{2n} at the point m ($T_m(M^{2n})^{\mathbb{C}}$) with following properties:*

i) $r^n(m)$ is a lagrangian subspace, i.e. $dim(r^n(m)) = n$ and isotropic ($[x, y] = 0, \forall x, y \in r^n(m)$),

ii) $r^n(m) \supset T_m(\Lambda^k)^{\mathbb{C}}$,

iii) $r^n(m)$ is disipative respect of $T_m(\Lambda^k)$, i.e.

$$\forall x \in r^n(m) \setminus T_m(\Lambda^k)^{\mathbb{C}} \quad holds \quad (1/2i)[x, \bar{x}] > 0, \tag{11}$$

iv) $r^n(m)$ is invariant respect the Hamiltonian flux $g^t_H, t \in \Re$ of a given function H,i.e. $\forall \in \Lambda^k, \forall t \in (-\infty, +\infty)$ holds

$$[(g^t_H)_{*,m}]^{\mathbb{C}}(r^n(m)) = r(g^t_H m), \tag{12}$$

where $[(g^t_H)_{*,m}]^{\mathbb{C}}$ is the complexification of the derivative at point m of an element g^t_H of Hamiltonian flux phases H associated to Hamilton function H.

Remark 10. *From now we omit the supraindex $\mathbb{C}$ that indicated the complexification, in the space and operator. But implicitly we use the properties enunciated previously.*

The purpose of this paper is to seek conditions for the existence and uniqueness of a Complex Germ on the a invariant tori to Hamiltonian system $(M^{2n},\omega^2,F_j),\ j=1,\ldots,k;(k<n)$.

4 Condition for the existence of the Complex Germ on the torus

Let

$$\Sigma=\{m\in M^{2n}:F_i(m)=f_i,i=1,\ldots,k,f=(f_1,\ldots,f_k)\in\Re^k\},\tag{13}$$

the intersection of the level surfaces defined by the functions $F_j, j=1,\ldots,k$ which contain the trajectories of the Hamiltonian system $x=JdF_j(x), j=1,\ldots,k$. Since $dF_1,\ldots,dF_k$ are linearly independent then Σ is a submanifold submerge in M^{2n} of dimension $2n-k$.

Let us define the action of additive group $\Re^k$ over M^{2n}, as follow: to each $\overrightarrow{t}=(t_1,\ldots,t_k)\in\Re^k$ correspond the difemeorphism of M^{2n} as:

$$g^{\overrightarrow{t}}=g_1^{t_1}o\ldots og_k^{t_k},\tag{14}$$

where $g_j=g_{F_j},\forall j=1,\ldots,k$. Since g_j^t are simplectic diffemeorphism the $g^{\overrightarrow{t}}$ as well, i.e. $(g^{\overrightarrow{t}})^*\omega^2=\omega^2,\forall\overrightarrow{t}\in\Re^k$.

4.1 Monodromy operator

The condition for existence of the Complex Germ are given in term of the monodromy operator which we defined in the following paragraph

Let us fix the point $m\in\Lambda^k$. Denoting by G the discrete subgroup of $\Re^k$ defined as follow

$$G=\{\overrightarrow{t}\in\Re^k:g^{\overrightarrow{t}}m=m\},\tag{15}$$

which dot not depend on the choice of the point m. The subgroup G can be generated by a set of k elements linearly independent(see [1]), i.e.

$$G=\{l_1T_1+\ldots,l_kT_k;\ l_i\ \in\Re;\ \ T_i\in G\ \text{ are linearly independent},\ \ i=1,\ldots,k\}.\tag{16}$$

Since Λ^k is invariant respect to Hamiltonian system $x'=JdF_j(x);j=1,\ldots,k$ we have $g_j^t\Lambda^k=\Lambda^k$ therefore $g^{\overrightarrow{t}}\Lambda^k=\Lambda^k$. We have also that the map $f:\Re^k\to M^{2n}$ given by $f(t)=g^{\overrightarrow{t}}m$ which to any $\overrightarrow{t}$ correspond a point in Λ^k is sobrejective, but is not injective because Λ^k is compact and $\Re^k$ is not; therefore there exist $t_1,t_2\in\Re^k;t_1\neq t_2$, such that $g^{t_1}m=g^{t_2}m$ and $g^{t_1-t_2}m=m$ with $t_1-t_2\in G$; which mean that the subgroup G is not trivial.

Definition 11. *Let $T\in G$. The operator $G_m=(g^T_{*,m}):T_m(M^{2n})\to T_m(M^{2n})$ is called monodromy operator of the subgroup G at point $m\in\Lambda^k$.*

Let us rewrite the monodromy operator in a way more amassing the express the sufficient condition for the existence of the Complex Germ. We gave

$$g^{T}_{*,m} = g^{l_1T_1+\ldots+l_kT_k}_{*,m}, \tag{17}$$

for certain $l = (l_1, \ldots, l_k) \in Z^k$, where $T_1, \ldots, T_k$ are the generator of subgroup G. Let denote $T_j = (t_{j1}, \ldots, t_{jk}); t_{ij}\Re; \forall i, j = 1, \ldots, k$. thus we obtain

$$g^{l_1T_1+\ldots+l_kT_k}_{*,m} = g^{l_1t_{11}+\ldots+l_kt_{k1}}_{1*,m} o \ldots o g^{l_1t_{1k}+\ldots+l_kT_{kk}}_{k*,m}. \tag{18}$$

Using that $g^{t+s}_j = g^t_j o g^s_j$ and $g^t_j o g^t_i = g^t_i o g^t_j$ and reordering (18) we obtain

$$g^{T}_{*,m} = (g^{l_1t_{11}}_{1*,m}+\ldots+g^{l_1t_{1k}}_{k*,m})o(g^{l_2t_{21}}_{1*,m}+\ldots+g^{l_2t_{2k}}_{k*,m})o\ldots o(g^{l_kt_{k1}}_{1*,m}+\ldots+g^{l_kt_{kk}}_{1*,m}), \tag{19}$$

or

$$G_m = G^{l_1}_1 o \ldots o G^{l_k}_k, \tag{20}$$

where $G_j = g^{T_j}_{*,m} : T_m(M^{2n}) \to T_m(M^{2n}), j = 1, \ldots, k$ is called monodromy operator with period T_j. Is valid the following

Lemma 12. *Over the tori Λ_k is possible to defined k vectorial field linearly independent.*

Proof. Since Λ^k is invariant respect to the k- Hamiltonian system $x^{'} = JdfF_j(x), j = 1, \ldots, k$, $JdF_j(m) \in T_m(\Gamma^k), \forall m \in \Gamma^k$. Also, as $dF_1, \ldots, dF_k$ are linearly independent in each point $m \in \Gamma^k$ and the operator J is regular we obtain that the vector $JdF_j(m), j = 1, \ldots, k$ are linearly in each point of $m \in \Gamma^k$. □

Note that $JdF_j, j = 1, \ldots, k$ in each point $m \in \Gamma^k$ constitute a basis of the tangent space $T_m(\Gamma^k)$ of Γ^k. Using the Γ^k is invariant we obtain

$$G_j(JdF_i) = \sum_{n=1}^{k} \beta_n JdF_n, \tag{21}$$

$$G_m(JdF_i) = \sum_{n=1}^{k} \mu_n JdF_n, \tag{22}$$

with $\beta_n, \mu_n \in \mathbb{C}; n = 1, \ldots, k$. Let $\Gamma_m = T_m(\Sigma)T_m(\Lambda^k)$; we have $dim(\Gamma_m) = 2(n-k)$.

It is valid the following

Lemma 13. *Let $[\theta] \in \Gamma_m$ then $G_j([\theta]) = [G_j(\theta)]$ and $G_m([\theta]) = [G_m(\theta)]$.*

Proof. Let $[\theta] = \{\theta : \theta^{'} \equiv \theta mod(T_m(\Lambda^k))\}$, $\theta^{'} \equiv \theta mod(T_m(\Lambda^k))$ if and only if $\theta^{'} = \theta + \sum_{i=1}^{k} \beta_i JdF_i$, where $\beta_i \in \mathbb{C}, i = 1, \ldots, k$, Also is valid that

$$G_j(\theta^{'}) = G_j(\theta) + \sum_{i=1}^{k} \mu_i JdF_i, \tag{23}$$

and by using equality (21) we have $G_j(\theta^{'}) = G_j(\theta) + \sum_{i=1}^{k} \mu_i JdF_i$ for certain $\mu_i \in \mathbb{C}, i = 1, \ldots, k$ and due to (20) also satisfy that

$$G_m(\theta^{'}) = G_m(\theta) + \sum_{i=1}^{k} \tau_i JdF_i, \tag{24}$$

$$\Xi_m : \Gamma_m \to \Gamma_m, \quad \text{such that} \quad [\theta] \to [G_m(\theta)], \tag{25}$$

$$\Xi_j : \Gamma_j \to \Gamma_j, \quad \text{such that} \quad [\theta] \to [G_j(\theta)], \tag{26}$$

with $j = 1, \ldots, k$. From now, the structure $[,]$ must be defined between equivalence class modulo $T_m(\Lambda)$ on each point $m \in \Lambda^k$. To do so, we need to prove that $[,]$ is compatible with respect to equivalence class i.e. if $\theta \equiv \sigma mod(T_m(\Lambda^k))$ and $\theta^{'} \equiv \sigma^{'} mod(T_m(\Lambda^k))$, then $[\theta, \theta^{'}] = [\sigma, \sigma^{'}]$. We assume by definition that $[[\theta], [\theta]] = [\theta, \theta]$.

we prove that $[\theta, \theta^{'}] = [\sigma, \sigma^{'}]$:

$$\theta = \sigma + \sum_{i=1}^{k} \alpha_i JdF_i, \quad \theta^{'} = \sigma^{'} + \sum_{i=1}^{k} \delta_i JdF_i. \tag{27}$$

By using bi-linearity property we have

$$[\theta, \theta^{'}] = [\sum_{i=1}^{k} \alpha_i JdF_i, \sum_{i=1}^{k} \delta_i JdF_i],$$

or

$$[\theta, \theta^{'}] = [\sigma, \sigma^{'}] + [\sum_{i=1}^{k} \alpha_i JdF_i, \sum_{i=1}^{k} \delta_i JdF_i], \tag{28}$$

$$+[\sigma, \sum_{i=1}^{k} \delta_i JdF_i] + [\sum_{i=1}^{k} \alpha_i JdF_i, \sigma^{'}] \tag{29}$$

we have

$$[\sigma, \sum_{i=1}^{k} \delta_i JdF_i] = \sum_{i=1}^{k} \delta_i [\sigma, JdF_i] = \sum_{i=1}^{k} \delta_i JdF_i(\sigma) = 0, \tag{30}$$

because $JdF_i(\sigma) = 0$ since $\sigma \in T_m(\Sigma)$. Analogously is prove that $[\sum_{i=1}^{k} \alpha_i JdF_i, \sigma^{'}] = 0$. Also that $[\sum_{i=1}^{k} \alpha_i JdF_i, \sum_{i=1}^{k} \delta_i JdF_i] = 0$ holds , since the functions $F_i, i = 1, \ldots, k$ stay involution. Finally we have the proof.

□

Definition 14. *The operator $\Xi_m : \Gamma_m \to \Gamma_m$ is called reduced monodromy operator at point m. i.e.,*

$$\Xi_m = (\Xi_1)^{l_1} o \dots o (\Xi_k)^{l_k}, \tag{31}$$

where the operators Ξ_j, $j = 1, \dots, k$ are called reduced monodromy operator with period T_j.

Also is valid the

Proposition 15. *The quotient space $\Gamma_m = T_m(\Sigma)/T_m(\Lambda^k)$ has a natural symplectic structure such that the operators Ξ_j, $j = 1, \dots, k$ are symplectic respect to the structure $[,]$ of the space $T_m(M^{2n})$ induce over this space.*

Proof. It is known that if the vectorial space V is given a bilinear, antisymmetric, degenerate form then over the quotient vectorial space $V/V^{\perp}$ is induced a bilinear, antisymmetric, no degenerate form. Taking $V = T_m(\Sigma)$ and the bilinear form $[,]$ which is degenerate over $T_m(\Sigma)$ and we use that $T_m(\Sigma)^{\perp} = T_m(\Lambda^k)$. Let $\sigma = \sum_{i=1}^{k} \sigma_i JdF_i$, which $\sigma_i \neq 0$ for some $i = 1, \dots, k$. Since $\sigma \in T_m(\Lambda^k)$ holds that $[x, \sigma] = 0,\ \ \forall x \in T_m(\Sigma)$ because $[x, IJdF_i] = dF_i(x) = 0$ (F_i is constant on Σ). This proved that $[,]$ is degenerate on $T_m(\Sigma)$ and $JdF_i \in T_m(\Sigma)$. Since $\dim(T_m(\Sigma)) = 2n - k$ then $\dim(T_m(\Sigma)^{\perp}) = k$ and $IJdF_i$ ($i = 1, \dots, k$) constitutes a basis of $T_m(\Lambda^k)$. Since $T_m(\Sigma)^{\perp} = T_m(\Lambda^k)$ we have that Γ_m is a subspace lineal symplectic. Now, we verify that the operator Ξ_j ($j = 1, \dots, k$) are symplectic. Let $[\theta], [\theta^{'}] \in \Gamma_m$. we have

$$[\Xi_j([\theta]), \Xi_j([\theta^{'}])] = [G_j([\theta]), G_j([\theta^{'}])],$$

where G_j, with $j = 1, \dots$ are the monodromy operators in Definition 11. From Lemma 13 we have

$$[\Xi_j([\theta]), \Xi_j([\theta^{'}])] = [[G_j(\theta)], [G_j([\theta^{'})]],$$

By definition of the product of two class we obtain

$$[\Xi_j([\theta]), \Xi_j([\theta^{'}])] = [G_j(\theta), G_j(\theta^{'})].$$

Since G_j are symplectic we have $[\Xi_j([\theta]), \Xi_j([\theta^{'}])] = [\theta, \theta^{'}]$. Using again the definition of the product of two class we have $[\Xi_j([\theta]), \Xi_j([\theta^{'}])] = [[\theta], [\theta^{'}]]$; so Ξ_j, with $j = 1, \dots, k$ are symplectic. □

Definition 16. *Two symplectic lineal operator $A_i : L^1 \to L^2$, with $i = 1, 2$. are called equivalent if there exist a lineal symplectic $\tau : L^1 \to L^2$ such that $A_2 = \tau A_2 \tau^{-1}$.*

For the purposes of this work we need to verify that the reduced monodromy operator F does not depend on the point $m \in T_m(\Lambda^k)$. They are precisely these operators which will serve to establish sufficient conditions for the existence of the complex germ. Notice Ξ_j, with $j = 1, \dots, k$ are symplectic operators then by Definition 14, operator Ξ_m in (31) is symplectic as well.

It is valid the following

Proposition 17. *Let $m, m^{'} \in \Lambda^k$ then operators Ξ_m and $\Xi_{m'}$ defined in* (31) *are equivalent.*

Proof. There exist $\overrightarrow{t} \in \Re^k$ such that $m = g^{\overrightarrow{t}} m^{'}$. Let the operator $G_{m,m'} = (g^{\overrightarrow{t}})_{*,m} : T_m(M^{2m} \to T_{m'}(M^{2m})$, which is symplectic (see Section 2). Also $G_{m,m'}(T_m(\Sigma)) = T_{m'}(\Sigma)$ and $G_{m,m'}(T_m(\Lambda^k)) = T_m(\Lambda^k)$ hold. Then we define the operator $\tau_{m,m'} : \Gamma_m \to \Gamma_{m'}$ by $\tau_{m,m'}([\theta]) = [G_{m,m'}(\theta)]$. We can check that $\Xi_{m'} \tau_{m,m'} = \tau_{m,m'} \Xi_m$. □

Remark 18. *Analogously it can prove that the operators Ξ_j, with $j = 1, \dots, k$ are equivalents at different points in the tori Λ^k.*

Let $\Pi : T_m(\Sigma) \to \Gamma_m$ the canonical map that each $\theta \in T_m(\Sigma)$ correspond its equivalence class $[\theta]$ modulo $T_m(\Lambda^k)$.

We have that is valid

Proposition 19. *It is valid that*

$$\Xi_j o \Pi = \Pi o G_j. \tag{32}$$

Proof. Let $\theta \in T_m(\Sigma)$; $\Pi(G_j(\theta)) = [G_j(\theta)]$. Besides that $\Xi_j(\Pi(\theta)) = \Xi_j([\theta]) = [G_j(\theta)]$, being proved the proposition. □

Definition 20. *A complex linear subspace $R \subset \mathbb{C}^n$ is called positive if $\forall x \in R; x \neq 0$ $(1/2i)[x, \bar{x}] > 0$ holds. It subspace is called negative if $(1/2i)[x, \bar{x}] < 0$.*

Lemma 21. *Let $R \subset \Gamma_m$ a lineal, positive, lagrangian and invariant respect to the operators Ξ_i, $i = 1, \dots, k$. then the subspace $r^n = \Pi^{-1}(R)$ is dissipative respect to $T_m(\Lambda^k)$, lagrangian and invariant respect of the operator G_i, $i = 1, \dots, k$, where Π is the canonical map.*

Proof. we have that $\Pi(x) = [x] mod T_m(\Lambda^k) = \{x + y : y \in T_m(\Lambda^k)\}$;

a) Let us prove that $r^n = \Pi^{-1}(R)$ is isotropic. Let $x_1, x_2 \in \Pi^{-1}(R)$ then $\Pi(x_1), \Pi(x_2) \in R; [\Pi(x_1), \Pi(x_2)] = 0$ for being r^n isotropic; by definition we have $[x_1, x_2] = [\Pi(x_1), \Pi(x_2)] = 0$, therefore r^n is isotropic.

b) Let us prove that r^n is lagrangian in $T_m(M^{2n}), i.e. dim(R) = (1/2) dim(\Gamma_m) = n - k, \Pi^{-1}(0) = T_m(\Lambda^k)$, then $dim(\Pi^{-1}(0)) = k$, also since $\Pi^{-1}(0) \subset \Pi^{-1}(R)$, then and $dim(r^n) = dim(Ker(\Pi)) + dim(Im(\Pi^{-1}(R))) = k + n - k = n$, then r^n is lagrangian.

c) Let us verify that r^n is dissipative respect to $T_m(\Lambda^k)$: we have that $T_m(\Lambda^k) \subset r^n$. Let $x \in r^n \setminus T_m(\Lambda^k)$ then $\Pi(x) = [x] \neq 0$. Besides we have

$[\bar{x}] = [\bar{x}]$, the dissipative condition of r^n is a consequence of the positivity of R, i.e.

$$(1/2i)[x, \bar{x}] = (1/2i)[[x], [\bar{x}]] = (1/2i)[[x], [\bar{x}]] = (1/2i)[\Pi(x), \Pi(x)] > 0,$$

due to R is positive.

d) The invariance of r^n respect of G_i $(i = 1, \dots, k)$ is a consequence of (32) and the invariance of R respect to Ξ_j $(i = 1, \dots, k)$. □

It is valid the following

Theorem 22. *Let $m \in \Lambda^k$ fixed. If there exist a lineal subspace N lagrangian, dissipative respect to $T_m(\Lambda^k)$, invariant respect all the operators G_i $(i = 1, \dots, k)$ then the exist a complex germ respect to the Hamiltonian system $x^{'} = JdF_j(x)$ $(j = 1, \dots, k)$.*

Proof. The idea is to construct a smooth map $r^n : m \to r^n(m)$ such $r^n(m) \subset T_m(M^{2n})$, $\forall m \in \Lambda^k$ satisfying Definition 9. Let us define $r^n(m) = N$. Since that map $f : \vec{t} \to g^{\vec{t}}m$, $\vec{t} \in \Re^k$ is subjective, we have that $\forall p \in \Lambda^k$, there exist $m \in \Lambda^k$ and $\vec{s} \in \Re^k$ such that $g^{\vec{s}}m = p$. Let us put

$$r^n(p) = (g^{\vec{s}})_{*,m}(r^n(m)). \tag{33}$$

Since the operator $(g^{\vec{s}})_{*,m}$ is symplectic and carries the subspace $T_m(\Lambda^k)$ in $T_p(\Lambda^k)$ then $r^n(p)$ $\forall p \in \Lambda^k$ is lagrangian, disipative with respect to $T_p(\Lambda^k)$.

Now, we verify that $r^n(m)$ is invariant: By Definition of Germ in (33) we have

$$(g^{\vec{t}})_{*,m}(r^n(m)) = (g^{\vec{t}})_{*,p} \o (g^{\vec{s}})_{*,m}(r^n(m)). \tag{34}$$

Using $g^{\vec{t}+\vec{s}} = g^{\vec{t}} \o g^{\vec{s}}$ in (34) we obtain

$$(g^{\vec{t}})_{*,p} \o (g^{\vec{s}})_{*,m}(r^n(m)) = (g^{\vec{t}+\vec{s}})_{*,m}(r^n(m)). \tag{35}$$

Using again (33) in (35), we obtain the invariance

$$(g^{\vec{t}+\vec{s}})_{*,m}(r^n(m)) = r^n(g^{\vec{t}+\vec{s}}m) = r^n(g^{\vec{t}} \o g^{\vec{s}}m) = r^n(g^{\vec{t}}p). \tag{36}$$

On the other hand, since $(g^0)_{*,m} = E_{2n}$, where E_{2n} is identity map, then by choosing appropriately the vector $\vec{t}$ we obtain $(g^{\vec{t}})_{*,m} = g_j^t m$, where $t \in \Re$ and g_j^t is the Hamiltonian flux associated to the function F_j, with $j = 1, \dots, k$. As a consequence the Germ is invariant respect to g_j^t. The smoothness of the Germ we proof in the Appendix A. □

Remark 23. *From Lemma 21 we have that is possible to construct the complex germ if the operators Ξ_i, $i = 1, \dots, k$ have a common positive, lagragian and invariant (P.L.I) linear subspace.*

Remark 24. *Proof of Theorem 22 consists in the construction of complex germ analogously as done in [10]. In this way, we proof the the map $r^n : m \to r^n(m)$ is smooth.*

Now following Remark 23, we find that sufficient conditions on operators Ξ_i, $i = 1, \dots, k$ such that they has a common P. L. I subspace.

Definition 25. *Let (M^{2n}, ω^2) a symplectic manifold. The subspace $L \subset T_m(M^{2n})$ in a point $m \in M^{2n}$ is simplectic if the restriction of ω^2 a L is no-degenerate.*

Definition 26. *A lineal transformation $SL_1 \to L_2$ between two lineal spaces is called stable if $\forall \epsilon > 0$ $\exists \delta > 0$ such that $|x| < \delta$ then $|S^n(x)| < \epsilon$ $\forall n \mathbb{N}, n > 0$ (see [2]).*

Proposition 27. *A symplectic map is stable if and only if all its eigenvalues belong to the unitary circle and S is diagonalizable.*

The proof can be found in [10], where also is proved that the stability condition of symplectic operator is equivalent to have a P. L. I subspace. With this idea, we present previous Lemmas used to prove sufficient condition the existence of a P.L.I subspace for the monodromy operators Ξ_i, $i = 1, \dots, k$.

Let A y B two stable operators. Let $K_1 = \{\sigma_1, \dots, \sigma_{2n}\}$ the eigenvalues of A, where 1 and -1 may also be included. Therefore, we can do the partition $K_1 = K_a \cup \{-1, 1\}$, where $K_a = \{\sigma_1, \dots, \sigma_r, \bar{\sigma_1}, \dots, \bar{\sigma_r} : Im\sigma_i \neq 0\}$ with $r <= n$ and σ_i distinct. Analogously for the operator B we have the partition $K_2 = K_b \cup \{-1, 1\}$, where $K_b = \{\mu_1, \dots, \mu_s, \bar{\mu_1}, \dots, \bar{\mu_s} : Im\mu_i \neq 0\}$ with $s <= n$ and μ_i distinct.

Let us denote by S_σ the subspace associated to the eigenvalue σ. In general the eigenvalues 1 and -1 they may not be included, these will be analyzed separately.

The following Lemma are valid

Lemma 28. *Let A y B two stable operators. Let us consider the restriction of the operator $A_j = A_{|S_{\sigma_j} \oplus S_{S_{\bar{\sigma}_j}}}$ and $B_j = B_{|S_{\sigma_j} \oplus S_{S_{\bar{\sigma}_j}}}$, where $\sigma_j \in K_a$. Then there exist a common P. L. I subspace for A_j and B_j.*

Proof. Let us consider the subspace $L = S_{\sigma_j} \oplus S_{S_{\bar{\sigma}_j}}$ which is symplectic and invariant for the operator A (see [10]), i.e. $AL = L$. We have that if p the multiplicity of $\sigma_j \in K_a$ then $\dim(L) = 2p$.

a)Now we prove that the subspace L is also invariant for the operator B. Since $A_j B = BA_j$ we have $A_j BL = BL$ therefore BL is invariant for the operator A_j and since B is a symplectic diffeomorphism then $\dim(BL) = 2p$.

Let $S_{\sigma_j} = \{A_j x = \sigma_j x\}$, then for $x \in S_{\sigma_j}$ we have $B(A_j(x)) = \sigma_j Bx$. Using $BA_j = A_j B$ we obtain $A_j(Bx) = \sigma_j B(x)$ therefore $Bx \in S_{\sigma_j}$ and as a consequence $BL \subset L$. Using that the operator B is a diffeomorphism we have $BL = L$ and the operator $B_j = B_{|L}$ is well defined.

b) Now since B is diagonalizable and symplectic operator in L, it is possible to obtain a descomposition through K_2 of L, i.e.

$$L = \sum_{i=1}^{d} (S_{\mu_j} \oplus S_{\bar{\mu_j}}) \oplus S_1 \oplus S_{-1}, \quad (37)$$

where S_1 and S_{-1} appear if 1 or -1 are eigenvalues of B in L. Let us consider the restriction $B_{j|L_{\mu_j}}$ of the operator B_j to $L_{\mu_j} = S_{\mu_j} \oplus S_{\bar{\mu_j}}$ then the following affirmation are true

1- $L_{\mu_j}, j = 1, \dots, d$ are symplectic operators and B_j are stable, therefore there exist a subspace P.L.I for $B_{j|L_{\mu_j}}$ in this subspace which we denote by R_{μ_j} (see [10] for the construction of this subpace).

2-The subspace S_1 is symplectic (see [10]). Also we have $B_{j|S_1} = Id_{|S_1}$, therefore all vector is eigenvalues for $B_{j|S_1}$. Then we can choose a collection of vector in S_1 such as they generated a P.L.I. subspace. Let us denote by R_o. Also satisfy $AR_o = R_o$. Analogously, for $B_{j|S_{-1}} = Id_{|S_{-1}}$ is possible to construct a P.L.I subspace denoted by R_{-1}.

Finally the subspace

$$R_\mu = R_o \oplus R_{-1} \oplus R_{\mu_1} \oplus \ldots \oplus R_{\mu_d}, \tag{38}$$

is a common P.L.I subspace to A_j and B_j in L. Thus, the proof is a consequence of the way that such subspace are constructed, i.e. it are positive and Lagrangian in L. Also, they are invariant for B_j,i.e. $B_j R_\mu = R_\mu$. So, we need to prove that is invariant for A_j, i.e. $A_j R_\mu = R_\mu$ for any $j = 1, \dots, d$.

The subspace R_{μ_j} is constructed from h eigenvector $e_1, \dots, e_h$ (h is a multiplicity of μ_j as eigenvalues of B_j in L), which are associated either μ_j or $\bar{\mu_j}$.

Let e_i an arbitrary with $i = 1, \dots, h$. Since $R_{\mu_1} \subset L$, there e_i is an eigenvalues of A associated to either σ_j or $\bar{\sigma_j}$. Assume that $A_j e_i = \sigma_j e_i$, therefore for $x \in R_{\mu_1}$, we have $A_j x \in R_{\mu_1}$.

□

Lemma 29. *Let A y B two stable operators. Let us consider the restriction of the operator $A_1 = A_{|S_\beta}$ and $B_1 = B_{|S_\beta}$, where β is either 1 or -1 which are eigenvalues of the operator A and B. Then there exist a common P. L. I subspace for A_1 and B_1.*

Proof. The subspace S_1 is symplectic (see [10]). Also we have $B_{|S_1} = Id_{|S_1}$, therefore all vector is eigenvalues for $B_{|S_1}$. Then we can choose a collection of vector in S_1 such as they generated a P.L.I. subspace. Let us denote by R_o. Also satisfy $AR_o = R_o$. Analogously, for $B_{|S_{-1}} = Id_{|S_{-1}}$ is possible to construct a P.L.I subspace denoted by R_{-1}. □

Due to the condition impose on these operators, we have that Lemma 31 is a Corollary of

Lemma 30. *Let A and B two stable symplectic operators that commutate in the symplectic space $\mathbb{C}^{2n}$, then they has a common P.L.I subspace.*

Proof. By proposition 27 the operators A and B have its eigenvalues in the unitary circle and are diagonalizable. Then we have $\mathbb{C}^{2n} = \sum_{i=1}^{d} (S_{\sigma_j} \oplus S_{\bar{\sigma_j}}) \oplus$

$S_1 \oplus S_{-1}$, where σ_j, $\bar{\sigma_j}$, 1 and -1 are eigenvalues of A (in general 1 and -1 are not necessarily eigenvalues of A). Now from Lemmas 28 and 29 we obtain that there exist a common P.L.I subspace for the A and B. $\square$

Lemma 31. *The reduce monodromy operators Ξ_i, $i = 1, \ldots, k$ are stable then they have a common P.L.I. subspace.*

Proof. The construction of the a common subspace for $k > 2$ is similar. After we have for two operators we construct for the rest. $\square$

4.2 Existence of the Germ

It is valid the following

Theorem 32. *If the reduce monodromy operators Ξ_i, $i = 1, \ldots, k$ are stable then there exist a complex germ invariant respect Hamiltonian system (M^{2n}, ω^2, F_j), with $j = 1, \ldots, k$ on the tori Λ^k.*

The proof is a consequence of Lemma 31 and Theorem 22.

Further we despited a necessary condition for the existence of the Germ.

Theorem 33. *Assume that there exist a complex germ on the isotropic tori Λ^k invariant respect to the Hamiltonian flow $x^{'} = IdF_j(x)$, with $j = 1, \ldots, k$. Then the reduce monodromy operators Ξ_i, $i = 1, \ldots, k$ are stable.*

Proof. We have $\forall m \in \Lambda^k$ there exist complex subspace lagrangian $r^n(m) \subset T_n(M^{2n})$ such that $r^n(m) \supset T_m(\Lambda^k)$ which is dissipative respect of $T_m(\Lambda^k)$, i.e. $\forall x \in r^n(m) \setminus T_m(\Lambda^k)$ we have $[x, \bar{x}]/2i > 0$ (condition (11) of Definition 9). Also, for $t \in \Re$ we have condition (12) of Definition 9. Therefore, the flow $g^{\vec{t}} = g^{t_1} o \ldots o g^{t_k} : M^{2n} \to M^{2n}$, where $t = (t_1, \ldots, t_n)\Re^k$ satisfy $g^{\vec{t}}_{*,m}(r^n(m)) = r^n(g^{\vec{t}} m)$. For $T \in G$ (G discrete subgroup of $\Re^k$ defined in) we have

$$g^T_{*,m}(r^n(m)) = r^n(g^T m),$$

which mean that $G_j(r^(m)) = r^n(m)$, where G_j, $j = 1, \ldots, k$ are the monodromy operators, which with period T_j, with $j = 1, \ldots, k$ generating the subgroup G. Let us consider the canonical projection map $\Pi : T_m(\Sigma) \to \Gamma_m = T_m(\Sigma)/T_m(\Lambda^k)$ and let $R = \Pi(r^n(m))$.

Now only rest to prove that the subspace R is lagragian, positive in Γ_m and invariant respect of the operators Ξ_j with $j = 1, \ldots, k$. Since $\Pi^{-1}(0) = T_m(\Lambda^k)$ we have $\dim(R) = \dim(r^n(m)) - k = n - k = (1/2)\dim(\Gamma_m)$. From the definition of symplectic structure in the quotient space Γ_m we obtain that R is isotropic and a consequence lagragian en Γ_m.

We have that if $\Pi(x) \neq 0$ then $x \notin T_m(\Lambda^k)$. Using that R is disipative respect of $T_m(\Lambda^k)$ and that $\bar{\Pi(x)} = \Pi(\bar{x})$ we have $(1/2i)[x, \bar{x}] = [\Pi(x), \Pi(\bar{x})] = [\Pi(x), \bar{Pi(x)}] > 0$ and as consequence R is positive.

Since $\Xi_j o \Pi = \Pi o G_j$, with $j = 1, \ldots, k$ follows that R is invariant respect of Ξ_j, with $j = 1, \ldots, k$. Thus we obtain a P.L.I subspace common for the operators Ξ_j, with $j = 1, \ldots, k$ therefore this operators are diagonalizables and its eigenvalues belong to unitary circle then by Proposition 27 are stables.

□

5 Uniqueness of the Germ

In this section we discuss about sufficient and necessary conditions of complex Germ. This issue is crucial since the germ is used in the construction of an asymptotic quase-classic it necessary to verify if obtained in this way is unique, which depend on the uniqueness of the germ. To display here a full characterization we summarized some basic concept.

Definition 34. *An stable map $S : L_1 \to L_2$ is called strong stable if all close map is stable.*

What means close map in above Definition?. Let consider the group of the symplectic map which constitute a submanifold of the lineal map of $\Re^n$. We consider any distance between two lineal map on $\Re^n$ as a distance between the respective matrix in a prefixed basis. i.e. Let $[s_{ij}]$ and $[s_{ij}^{'}]$ the matricial representantin of two lineal map $L_1, L_2 : L^1 \to L^2$ then they are close if $\max |s_{ij} - s_{ij}^{'}| \leq \epsilon \ \forall \epsilon > 0$ and $i, j = 1, \ldots, k$.

Let $A : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ an symplectic operator and σ an eigenvalues of A. We denote by L_σ the maximal invariant subspace respect to A associated to σ.

Definition 35. *The eigenvalues σ is called elliptic positive (negative) if the subspace L_σ is positive (negative).*

In [10] a collection of results of the elliptic eigenvalues are obtained. Some og then we summarized here

Proposition 36. *An symplectic map $L : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ is strong stable if all its eigenvalues are elliptic belong to the unitary circle.*

Also is valid

Proposition 37. *An symplectic map $L : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ is strong stable if has a unique P.L.I subspace.*

The following Lemma holds

Lemma 38. *If there exist a unique common subspace P.L.I to the reduce monodromy operatos Ξ_j, with $j = 1, \ldots, k$ then there exist a unique complex germ on the tori Λ^k.*

Proof. Assume that there exist two distinct complex germ r_1^n and r_2^n, i.e. there exist a point m_o such that $r_1^n(m_o) \neq r_2^n(m_o)$. Let Π the canonical sobrejection and let the subspace $R_j = \Pi(r_j(m_o))$, $j = 1, 2$ which are by definition lagragian and positive in Γ_{m_o} and invariant respect the operators Ξ_j, with $j = 1, \ldots, k$. By Analogous process to describe in Lemma 28 is possible to construct more that one P.L.I subspace common to the operators Ξ_j, with $j = 1, \ldots, k$. □

Now we give sufficient condition for the existence of a unique complex germ.

Theorem 39. *In the reduce operator of monodromy are stable and there exist a least one strong stable then there exist a unique complex Germ on the tori Λ^k invariant respect to the Hamiltonian system (M^{2n}, ω^2, F_j) with $j = 1, \dots, k$.*

Proof. Let Ξ_j for some $j \in J = 1, \dots, k$ a strong stable operators. By Proposition 37 there exist a unique P.L.I subspace for Ξ_j which we denote by R. As the operator Ξ_i, with $i \in I \diagdown j$ commute with Ξ_j then by Lemma 31 is possible to construct a common P.L.I subspace for these operators, which is a unique. Then by Lemma 38 we have a Theorem. □

A necessary condition for the existence of germ is given the following

Lemma 40. *If there exist a unique complex germ on the tori Λ^k invariant respect to the Hamiltonian system (M^{2n}, ω^2, F_j) with $j = 1, \dots, k$. Then there exist a unique common P.L.I subspace for the reduce monodromy operators Ξ_j, with $j = 1, \dots, k$.*

Proof. Let the stable reduce monodromy operators Ξ_j, with $j = 1, \dots, k$. Then by Lemma 31 these operators have for a common P.L.I subspace $R \subset \Gamma_m$ for each $m \in \Lambda^k$.

From Theorem 22 with this subspace we can construct the complex germ $r^n = r^n(R)$. Besides, to distinct subspace correspond distinct Germ, i.e. if $R_1 \neq R_2$ then for $r_1^n = r^n(R_1)$ and $r_2^n = r^n(R_2)$ we have $\forall m \in \Lambda^k$ that $r_1^n(m) \neq r_2^n(m)$. Assume that there exist $m \in \Lambda^k$ such $r_1^n(m) = r_2^n(m)$; let $m^{'} \in \Lambda^k$ which serves to construct the reduce operator $\Xi_{m'}$ and the subspace R_1 and R_2. Since there exist $\vec{s} \in \Re^k$ such that $g^{\vec{s}}(m^{'}) = m$, and if we define $r_1(m) = g^{\vec{s}}_{*,m'}(r^n(m^{'}))$; $r_1^n(m^{'}) = R_1$ and $r_2^n(m) = g^{\vec{s}}_{*,m'}(r^n(m^{'}))$; $r^n(m^{'}) = R_2$ we obtain a contradiction $R_1 = R_2$. □

We have the following

Lemma 41. *In there is a unique P.L.I subspace common to the reduce monodromy operators Ξ_j, with $j = 1, \dots, k$ then these operators are stable and a least one is strong stable.*

Proof. We assume that all the operators Ξ_j, with $j = 1, \dots, k$ are stable and none is strong stable then from Lemmas 28, 29 it is possible to construct more of one P.L.I subspace common for two operators, i.e. R_1 and R_2. With the procedure of Lemma 31 is possible to construct more of one common P.L.I for the operators Ξ_j, with $j = 1, \dots, k$. □

We present a necessary condition for the complex germ

Theorem 42. *If there is a unique complex germ invariant respect to the Hamiltonian system (M^{2n}, ω^2, F_j) with $j = 1, \dots, k$. Then the reduce monodromy operators Ξ_j, with $j = 1, \dots, k$ are stable and a least one is strong stable.*

The proof follow from the Lemmas 38 and 40.

6 Smoothness of the complex Germ

A p-distribution on a manifold M, with $p \leq dim(M)$ is a map such that to each point $m \in M$ correspond a subspace p-dimensional $\theta(m)$ of the $T_m(M)$. We say that such distribution is smooth if there are p smooth vectorial fields $X_1, \ldots, X_k$ defined on a neighbor U of the point m such that $X_1(m), \ldots, X_k(m)$ generate the subspace $\theta(m)$.

Since the complex Germ is a n-dimensional distribution with certain additional properties (see Definition) the smoothness property is a consequence of the own construction of the germ. From () the germ is generate by a P.L.I subspace R. We assume that this subspace is generated by the vector $r_1, \ldots, r_n$. Since the map $(g^t_{F_j})_{*,m}$ are smooth the the vectorial field X_j, $j = 1, \ldots, k$ defined in each point $m \in \Lambda^k$ as

$$X_j(m) = (g^{\vec{t}})_{*,m_o} r_j, \tag{39}$$

where $\vec{t} \in \Re^k$ satisfy $g^{\vec{t}} m_o = m_o$, with $g^{\vec{t}} = g_1^{t_1} o \ldots, g_k^{t_k}$ and $g_j^{t_1}$ are Hamiltonian flows associated to the Hamilton functions F_j. It is possible to check that X_j, $j = 1, \ldots, k$ are smooth and generated the germ $r^n(m)$.

7 Application to the Hamiltonian system with cyclic variables

In this section we study a $2n$-dimensional symplectic manifold that contains a 2k-dimensional manifold ($k < n$) which consist of k invariant isotropic tori respect to certain Hamiltonian system. These system in the local coordinate (I, p, θ, q) has the form $H = H(I, p, q)$ and the rest are $k-1$ coordinates. These are called system with cyclic variables. The basis and the problem are described following. Let (M^{2n}, ω^2) a symplectic manifold with canonical variables p, q..

Definition 43. *An atlas on the manifold (M^{2n}, ω^2) is called symplectic if the the coordinate space $(p, q) in \Re^{2n}$ the symplectic structure take the form $\omega^2 = \sum_{i=1}^{n} dp_i \wedge dq_i$ and the cart $\Phi o \Theta^{-1} : \Theta(U_1 \cap U_2) \rightarrow \Phi(W_1 \cap W_2)$ are symplectic transformations, where $\Phi : U_1 \rightarrow W_1$, $\Theta : U_2 \rightarrow W_2$, with $U_1, U_2 \subset M^{2n}$ and $W_1, W_2 \subset \Re^{2n}$,*

Definition 44. *A Hamiltonian system (M^{2n}, w^2, H) with Hamilton function H is called system with k cyclic variables if there are the system of symplectic coordinates I, p, θ, q such that $H = H(I, p, q)$. The cyclic variables do not appear in the expression of the Hamilton function which we denote by $\theta = (\theta_1, \ldots, \theta_k)$.*

Let the variables $I = (I_1, \ldots, I_k)$, $p = (p_1, \ldots, k)$ canonical conjugate of the variables $\theta = (\theta_1, \ldots, \theta_k) mod 2\pi$, $q = (q_1, \ldots, q_k)$ with $1 <= k <= n$. i.e. the phase space M^{2n} have the coordinate system (I,p,θ,q) where the symplectic structure w^2 can be written in the form

$$w^2 = \sum_{i=1}^{k} dI_i \wedge \theta_i + \sum_{i=1}^{k} dp_i \wedge q_i, \tag{40}$$

We assume that coordinate system define a diffeormorphism from M^{2n} under the space consist of the direct product of the tori T^k with open region of $\Re^{2n-k}$. In this system the canonical equation of the Hamilton function take the form

$$I^{'} = 0, \quad p^{'} = -H_q, \theta^{'} = H_I, \quad q^{'} = H_p, \tag{41}$$

where $H_q = (\partial H/\partial q_1, \ldots, \partial H/\partial q_k$ and $H_p = (\partial H/\partial p_1, \ldots, \partial H/\partial p_k$.

We consider the subset

$$N = \{(I,p,\theta,q) : H_p = H_q = 0\}, \tag{42}$$

and we assume that is connected. Let us denote by $\Lambda^k = \Lambda^k(I_o,p_o,q_o)$ the k-dimensional tori defined by the the equalities $\{(I,p,\theta,q) : I = I_o, p = p_o, q = q_o\}$. It is possible to check that N represent the union of such tori.

We consider the restriction of the vectorial Hamiltonian fiel to the tori Λ^k where a trajectory on such field take the form $g_H^t = (I_o,p_o,H_I t+\theta,q_o)$ defined for all $t \in (-\infty,+\infty)$ because Λ^k is compact. It is possible to check that any tori $\Lambda \subset N$ is invariant respect to this system. Also, from the symplectic structure in 40 we deduce that $\omega^2|_{\Lambda^k} = 0$, i.e. the tori are isotropic. Therefore H is the union of isotropic, invariant respect to the Hamiltonian system (M^{2n},ω^2,H).

We have the following Definition

Definition 45. *We say that F is first integral of the Hamiltonian system with function H if the Poisson braked satisfy* $[F,H]$*=0.*

It is possible to check that described above the coordinate function $I_1,\ldots,I_k$ are first integral of the system (M^{2n},w^2,H).

We have

Proposition 46. *Assume that for any point in N given in* (42) *the determinant of* $Hess(H)$ *is distinct of zero , i.e.*

$$\mathrm{H_{pp}}\,\mathrm{H_{qq}} - \mathrm{H_{qp}}\,\mathrm{H_{pq}} \neq 0. \tag{43}$$

Then N is sub-manifold symplectic 2k-dimensional that has locally the form of a map $(I,\theta) \to (p,q)$ *that does not depend on the coordinate* θ*, i.e.* $p = P(I), q = q(Q)$.

The proof of this result can be found in [10].

From now, we assume that Λ^k consist of regular points, i.e. $H_I \neq 0$, as a consequence the functions $H, I2, \ldots, I_k$ are k first integral involution. The problem is under what conditions there is a complex Germ on Λ^k with respect to the Hamiltonian system defined by Hamilton functions $H, I2, \ldots, I_k$.

The reduce monodromy operators for this case are the following: $G1 = (g_H^t)_{*,m}, G_j = (g_{I_j}^t)_{*,m} = E_{2n}$, with $j = 2, \ldots, k$ and E_{2n} is identity operator of order $2n$. This mean that the invariance condition if only necessary to check for the operator G_1. Applying the result of this paper we obtain the sufficient and necessary for the existence of Germ different from those obtained in those obtained in [2].

8 Conclusion

In this work we give answer to the question about the existence and uniqueness of a complex germ on a isotropic tori invariant respect to the Hamiltonian flows defined by k function $F_1, \ldots, F_k$ that stay involution in the phase state M^{2n}.

We proof that there exist such germ if and only if the reduce monodromy operator with period T_j, $j = 1, \ldots, k$ are stable. This germ is unique if at least one operator is strong stable.

The result obtained here were applied to the case of an Hamiltonian with cyclic variables resulting in new condition for the existence and uniqueness of complex germ.

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