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Analytic regularization of an inverse filtration problem in porous media

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Abstract

In Alvarez *et al* (2006 *Inverse Problems* **22** 69–88), we studied the direct and inverse problem of deep bed filtration for a model proposed by Herzig *et al* (1970 *Indus. Eng. Chem.* **65** 8–35), which describes particle retention in a porous medium under injection of water with solid inclusions. However, many questions were left unanswered. Here most of these issues are solved for an alternate model of deep bed filtration also from [10]. Both models depend on the filtration function, which must be recovered from experimental data histories by solving an inverse problem. The main issues solved here are: (i) we establish the stability of the inverse problem under adequate conditions, leading to a robust numerical procedure; (ii) we present a numerical method for the direct problem that allows us to solve the inverse problem by parameter estimation in several spatial dimensions. We impose conditions on the experimental data preprocessing that allow us to recover analytic filtration functions with adequate physical behavior.

(Some figures may appear in colour only in the online journal)

1. Introduction

Deep bed filtration occurs in porous media undergoing injection of water containing solid particles, with volumetric concentration c. Particles in the water are retained at the pores at a rate λc, i.e. the rate is proportional to the concentration of the suspended particles. Here λ is called filtration coefficient. This quantity cannot be measured directly. Some simple but useful models assume that λ is constant. Methods for determining the constant filtration coefficient from the histories of the injected particle concentration at the inlet of the rock cylinder and of the effluent particle concentration at the outlet were studied in [19, 21, 9, 7] and [2]. Other models consider that λ depends on σ, the concentration of the particles already retained, i.e. $\lambda = \lambda(\sigma)$, the filtration function. Depending on the application, this function is often positive and monotone non-increasing. This is the case for the class of problems solved in this work. A general method presented in [2] determines $\lambda(\sigma)$ based on the effluent and injected particle concentration histories. It involves an inverse problem that is solved by a functional equation derived from an invariant along characteristic lines for the particle transport equation.

Some issues concerning the above-mentioned inverse solution method were left unresolved in [2], as the numerical method proposed therein often generates unphysical oscillatory $\lambda(\sigma)$, even when the input data histories are approximated by a smooth function. One issue is whether it is possible to obtain a monotone decreasing filtration function from a class of experimental data histories. In [2], we found that analyticity of the data reduces oscillations in the resulting analytic filtration function. A further issue is that the stability of the inverse problem was obtained based on the properties of the filtration function rather than the properties of the data.

Another issue concerning the numerical solution of the direct problem was also left unresolved in [2]. In that work, we developed a finite difference numerical method for advancing the evolution problem in time. The method was slow when using it in combination with an optimization procedure for parameter estimation as a method for solving the inverse problem involving long effluent concentration histories. (However, it did work well for short histories, especially in combination with Tichonov regularization [1, 3].) In order to circumvent this slowness, here we have modified the model in a way that will be explained later, allowing us to use an implicit box finite difference scheme which is very fast. Also, the numerical method for the direct problem developed in [2] is not easy to generalize efficiently to radial or spherical coordinates, which are important in petroleum applications. The current scheme for the direct method does not have this limitation.

Any stability condition for the inverse problem implies restrictions on the input data. The stability condition presented in [2] was rather weak, as it also required *a priori* information on the output of the inverse problem, i.e. on the filtration function. Here we present a stronger stability result, which imposes further conditions on the input data, but requires no *a priori* information on the filtration function. The new result is based on the intrinsic properties of the functional equation used in the inverse problem.

Besides the stability of the inverse problem, it is important to ensure that the filtration function is monotone decreasing for an appropriate class of data. In this work, we present conditions on the approximate input data that guarantee such monotonicity.

We impose analyticity, stability and monotonicity in the inverse problem by requiring that experimental data points be approximated by an analytic function that satisfies the above-mentioned conditions. This is the preprocessing step in the methodology we propose here. The procedure for solving the inverse problem as applied to the preprocessed data yields a filtration function that has all desired properties. The combination of preprocessing the experimental data and resolution of the inverse problem is a more robust numerical procedure than the one proposed in [2]. This procedure is the main practical result in this work.

In their fundamental work [10], Herzig *et al* proposed two models for filtration. They are equally valid in practice. One of them was analyzed in [2]. The other one is analyzed here. We prefer this one as the numerical solution for the direct problem has fewer limitations; this is important when optimization is used. Most of the properties of the inverse problems are identical for the two models.

The paper is organized as follows. In section 2, we present the deep bed filtration model as a quasi-linear system of hyperbolic equations with source and sink terms containing the filtration function $\lambda(\sigma)$. For simplicity, the injected particle concentration is constant. We also present a finite difference scheme for solving the evolution problem; the method generalizes to higher spatial dimensions, so it is more useful than the method presented in [3], which is based on the system of ordinary differential equations (2.9*a*). Another advantage of this method is that it is pointwise implicit, so that it does not suffer from spurious CFL limitations. In section 3, we summarize a recovery method for $\lambda(\sigma)$ utilizing a functional equation. In section 4, we present conditions on the approximate effluent concentration history that lead to the monotonicity of $\lambda(\sigma)$. In section 5, we prove that $\lambda(\sigma)$ is analytic, assuming that the approximate effluent concentration history satisfies certain properties, including analyticity. A condition for the stability of the inverse problem is obtained in section 6 as a consequence of the intrinsic properties of the functional equation. Numerical experiments are presented in section 7, where we see good agreement between experimental effluent concentrations data and the effluent concentration histories based on the recovered filtration functions, except in a case when full preprocessing is impossible. These histories were obtained using the direct solver in section 2. Also, we show through some examples how the sufficient condition for stability in section 6 and monotonicity in section 4 help in constructing the new robust numerical recovery procedure. We draw some conclusions in section 8.

2. The direct problem

Our work utilizes a model for deep bed filtration proposed in the fundamental work of Herzig *et al* [10], which consists of equations expressing the particle mass conservation and the particle retention process ([8, 10, 16]). They form a quasi-linear hyperbolic system of equations containing the empirical filtration function $\lambda(\sigma)$, which represents the kinetics of particle retention. This model for linear flow is given in non-dimensional form by the system:

$$\frac{\partial \sigma}{\partial T} + \frac{\partial c}{\partial X} = 0, \tag{2.1}$$

$$\frac{\partial \sigma}{\partial T} = \lambda(\sigma)c. \tag{2.2}$$

Equation (2.1) was called 'first approximation' by Herzig *et al* [10]. The physical domain is dimensionless position $X \in [0, 1]$ and time $T \geqslant 0$. The non-dimensional time 'unit' is called PVI, from 'pore volume injected'. The unknowns $c(X, T)$ and $\sigma(X, T)$ are the suspended and deposited particle concentrations, respectively. As boundary data, we assume that the suspended solid particle concentration entering the porous medium is given and constant, i.e.

$$c(0, T) = c_o > 0, \quad T \geqslant 0. \tag{2.3}$$

Here we have taken the inlet concentration $c(0, T)$ as constant in (2.3) just for simplicity. The general case for variable inlet concentration data is studied in [1] and [2]; the results from the current work extend directly to the general case. As initial data, we assume that the rock contains no deposited particles:

$$\sigma(X, 0) = 0, \quad 0 \leqslant X \leqslant 1, \tag{2.4}$$

The direct problem of determining $c(X, T)$ and $\sigma(X, T)$ given $\lambda(\sigma)$ for equations (2.1)–(2.4) with (2.1) replaced by

$$\frac{\partial}{\partial T}(c + \sigma) + \frac{\partial c}{\partial X} = 0 \tag{2.5}$$

was also proposed by Herzig *et al* [10], who called it the 'second approximation'. This model was studied in [2]; see also [3].

In real applications, the suspended particle concentration c varies slowly compared to the deposited particle concentration σ. Thus, the time derivatives of c can be dropped from the mass conservation equation (2.5), leading to (2.1). This modification does not introduce a relevant change in the solution [10], except initially, but it greatly facilitates the numerical solution of the direct evolution problem [20], it makes no difference as to the inverse problem. (Thus essentially all properties of the inverse problem stated here apply to the inverse problem in [2]).

The existence and well posedness of the direct problem (2.1) and (2.2) can be established as in [2] under the following.

Assumption 2.1. *The filtration* $\lambda(\sigma)$ *is a positive* C^1 *function in* $0 \leqslant \sigma < 1$.

From equations (2.2) and (2.3), we obtain the following ordinary differential equation for $\sigma(0, T)$ along the line $X = 0$:

$$\frac{d}{dT}\sigma(0, T) = \lambda(\sigma(0, T))c_o, \quad \text{with} \quad \sigma(0, 0) = 0. \tag{2.6}$$

Integrating equation (2.6) provides $\sigma(0, T)$, which is always positive and increasing and C^1 (by assumption 2.1). From equations (2.1) and (2.2), we obtain the following ordinary differential equation for $c(X, 0)$ along the line $T = 0$:

$$\frac{d}{dX}c(X, 0) = -\lambda(\sigma(X, 0))c(X, 0), \quad \text{with} \quad c(0, 0) = c_o. \tag{2.7}$$

Using the assumption given in (2.4) in equation (2.7) leads to

$$c(X, 0) = c_o \exp(-\lambda(0)X). \tag{2.8}$$

The proof of the following result is similar to the proof of theorem 2.4 in [2].

Theorem 2.2. *Under assumption 2.1, there exists a unique, well-posed solution in* $R = \{(X, T) : 0 \leqslant X \leqslant 1; T \geqslant 0\}$ *for the system (2.1) and (2.2) with initial data (2.4) and boundary data (2.3). This solution is* C^2 *in an open set containing R; it is obtained by solving the system of ODEs:*

$$\frac{d\sigma}{dX} = -\lambda(\sigma)\sigma \tag{2.9a}$$

$$\frac{dc}{dX} = -\lambda(\sigma)c, \tag{2.9b}$$

for each T*, with initial conditions for* $c(0, T)$ *and* $\sigma(0, T)$ *as given in (2.3) and calculated in (2.6), respectively.*

Now we present a result that was stated first in [10].

Lemma 2.3. *Consider the solution of (2.1)–(2.4) under assumption 2.1. Then*

$$\frac{\sigma(X, T)}{c(X, T)} = \frac{\sigma(0, T)}{c(0, T)}, \quad \text{for} \quad T \geqslant 0. \tag{2.10}$$

Proof. From equations (2.9*a*) and (2.9*b*), for constant T we obtain $d\sigma/dc = \sigma/c$. Integrating this equation, we see that σ/c is invariant along lines $T = \text{const}$, hence (2.10). □

Remark 2.4. Since the RHS of (2.9*b*) is negative, the function $c(1, T)$ is C^2 and $c(1, T) < c_o$ in some time interval $[0, A]$, see [1].

As an example, the solution of (2.1) and (2.2) for constant filtration function $\lambda(\sigma) = \lambda_0$ is

$$c(X,T) = c_{io}\mathrm{e}^{-\lambda_0 X}, \quad \sigma(X,T) = \lambda_0 c_{io} T \mathrm{e}^{-\lambda_0 X}; \quad 0 \leqslant X \leqslant 1. \tag{2.11}$$

For medium and large times (*e.g.*, $T > 10$), it is easy to verify that this solution is almost the same as the solution of the model that includes the accumulation term $\partial c/\partial T$ on equation (2.5) used in [7] and [2], which we write for the convenience of the reader:

$$c(X,T) = c_{io}\mathrm{e}^{-\lambda_0 X}, \quad \sigma(X,T) = \lambda_0 c_{io}(T - X)\mathrm{e}^{-\lambda_0 X}; \quad \text{for} \quad T > X, \quad 0 \leqslant X \leqslant 1, \tag{2.12}$$

and 0 otherwise. Here, we imposed the typical initial condition: $c(x,0) = 0$. Note that the relative discrepancy between (2.11) and (2.12) is bounded by constant times $1/T$. We expect the two models to give almost identical results for any filtration function $\lambda(\sigma)$ except for short times, which are not relevant in practice. Here we focus on the system (2.1) and (2.2) because it has numerical advantages when solving the direct problem, as we explain now (see also [20]).

2.1. *Numerical method for the direct problem*

The method proposed in theorem 2.2 to obtain the solution of the direct problem has theoretical value and it is useful for proving that the direct problem is well posed. A numerical method based directly on (2.9) can be found in [3]. However, from the practical point of view it is not very useful and does not allow generalization to problems in higher spatial dimensions. We propose here another method that works well and can be generalized for these other cases. Note that the system of equations (2.1) and (2.2) can be rewritten as

$$\frac{\partial c}{\partial X} = -\Lambda(\sigma, c), \tag{2.13}$$

$$\frac{\partial \sigma}{\partial T} = \Lambda(\sigma, c), \tag{2.14}$$

where $\Lambda(\sigma, c) = \lambda(\sigma)c$.

Let us denote by $\{x_m = mh, m = 0, 1, 2 \ldots, M\}$ a discretization of the interval $[0, L]$ in the spatial axis, for h defined as L/M. Similarly, we denote by $\{t_n = nk, n = 0, 1, 2 \ldots, A\}$ a discretization of the interval $[0, A]$ in the time axis, for k defined as A/N. Let us use the notation $c_m^n = c(x_m, t_n)$ and $\sigma_m^n = c(x_m, t_n)$. We can solve numerically the system of equations (2.13), by means of trapezoidal scheme (see [20]):

$$\frac{c_{m+1}^{n+1} - c_m^{n+1}}{h} = -\frac{1}{2}\left(\Lambda_{m+1}^{n+1} + \Lambda_m^{n+1}\right), \tag{2.15}$$

$$\frac{\sigma_{m+1}^{n+1} - \sigma_{m+1}^{n}}{k} = +\frac{1}{2}\left(\Lambda_{m+1}^{n+1} + \Lambda_{m+1}^{n}\right). \tag{2.16}$$

In [20], it was proved that if $\Lambda(c, \sigma)$ is C^2 then the scheme (2.15) and (2.16) has second-order accuracy in space and time. Note that the unknowns c_{m+1}^{n+1} and σ_{m+1}^{n+1} appear implicitly in Λ_{m+1}^{n+1}, so we need to solve (2.13)–(2.16) by using Newton's method in each box.

2.1.1. Boundary and initial conditions. To solve (2.13) and (2.14) numerically, we need the values of σ and c on the boundary. On the axis $X = 0$, we assume that $c(0, T) = c_o$ is given and $\sigma(0, T)$ is obtained by solving the ordinary differential equation (2.6). On the axis $T = 0$, we take $\sigma(X, 0) = 0$ and $c(X, 0)$ is obtained from (2.8).

Table 1. Maximum absolute error between analytic and predicted deposited and suspended concentrations appear in columns four and five respectively. In this numerical example, we take $T \in [0, 0.5]$. Here $\lambda(\sigma) = 2 - 3\sigma$, for $T < 0.5$ we have that $\sigma < 3/2$, so $\lambda(\sigma)$ remains positive and C^1.

h	k	k/h	Max. absolute error in σ	Max. absolute error in c
0.25	0.0057	0.22	$7.3801\ 10^{-7}$	$7.6614\ 10^{-7}$
0.125	0.0032	0.2564	$1.8499\ 10^{-7}$	$1.9155\ 10^{-7}$
0.0063	0.0016	0.2564	$4.6322\ 10^{-8}$	$4.7893\ 10^{-8}$

2.1.2. Numerical examples. Now, we compare the result of the scheme (2.15) and (2.16) utilizing an explicit solution when $\Lambda(\sigma, c) = \lambda(\sigma)c$, for linear filtration function (with $a \geqslant \lambda_0$) (see [9, 4]), i.e.

$$\lambda(\sigma) = \lambda_0 - a\sigma, \quad \text{for} \quad 0 \leqslant \sigma < \lambda_0/a \tag{2.17}$$

and

$$\lambda(\sigma) = 0 \quad \text{for} \quad \lambda_0/a \leqslant \sigma \leqslant 1, \tag{2.18}$$

or $\lambda(\sigma) = \max\{\lambda_0 - a\sigma, 0\}$. For this case, the solution of the system (2.13) and (2.14) is

$$\sigma(X, T) = \frac{\lambda_0}{a}\left[1 + \frac{e^{-c_{i0}aT}\, e^{\lambda_0 X}}{1 - e^{-c_{i0}aT}}\right]^{-1} \text{ for } T > 0 \text{ and } \sigma(X, 0) = 0 \text{ for } T = 0, \tag{2.19}$$

$$c(X, T) = \frac{c_{i0}a\sigma(X, T)}{\lambda_0(1 - e^{-c_{i0}aT})} \text{ for } T > 0, \text{ and } c(X, 0) = c_{io}e^{-\lambda_0 X} \text{ for } T = 0. \tag{2.20}$$

In table 1, we show the maximum absolute error between the numerical solution of (2.15) and (2.16) and the exact solution in (2.19) and (2.20) for three mesh sizes. One can see that when the mesh size is divided by 2 the maximum absolute error reduces to one fourth, in agreement with the fact that the numerical scheme is second order accurate in time and space.

Therefore, the scheme (2.15) and (2.16) represents a reliable numerical method for solving the direct problem.

This method can also be used as an efficient direct solver to recover the filtration function by using an optimization algorithm as in [17].

Remark 2.5. In the case where the filtration function is linear, as in equations (2.17) and (2.18), the solution for the system(2.2)–(2.5) and $c(x, 0) = 0$ is (see [1])

$$\sigma(X, T) = \frac{\lambda_0}{a}\left[1 + \frac{e^{-c_{i0}a(T-X)}\, e^{\lambda_0 X}}{1 - e^{-c_{i0}a(T-X)}}\right]^{-1} \quad \text{and} \quad c(X, T) = \frac{c_{i0}a\sigma(X, T)}{\lambda_0(1 - e^{-c_{i0}a(T-X)})}, \tag{2.21}$$

when $T > X$ and 0 otherwise. Comparing the solution (2.21) with the solution of the system (2.1)–(2.4) shown in equations (2.19) and (2.20) one may see good agreement for medium and large times.

3. The functional equation

Here we summarize a recovery method for the filtration function analogous to that in [1] and [2]. We assume that the effluent concentration $c(1, T)$ is a given C^2 function of T and that c and σ satisfy (2.1)–(2.3) and (2.10). We introduce the C^3 function in $0 \leqslant T \leqslant A$

$$C(T) \equiv \int_0^T c(1, s)\, ds. \tag{3.1}$$

We now obtain relationships between the deposited and suspended particle concentrations at the inlet and outlet points. From assumption 2.1, we can define the first integral Ψ of $1/\lambda$ and the quantity δ

$$\Psi(\sigma) = \int_0^\sigma \frac{\mathrm{d}\eta}{\lambda(\eta)}, \qquad \delta = \int_0^1 \frac{\mathrm{d}\eta}{\lambda(\eta)}. \tag{3.2}$$

Note that $\Psi(0) = 0$. Depending on the behavior of $\lambda(\sigma)$ near 1, the range $\Psi : [0, 1) \to [0, \delta)$ is either a finite or infinite interval. From (2.2), it follows [2] that

$$\frac{\partial \Psi(\sigma)}{\partial T} = c, \quad \text{for } \sigma \in [0, 1]. \tag{3.3}$$

We integrate equation (3.3) in T, then use equations (2.4) to reveal that $\sigma(0, 0) = \sigma(1, 0) = 0$. Setting $X = 0$ and $X = 1$ in the integrated equations, and using (3.1) we obtain

$$\Psi(\sigma(0, T)) = c_o T, \tag{3.4a}$$

$$\Psi(\sigma(1, T)) = C(T). \tag{3.4b}$$

From assumption 2.1 and the definition of Ψ in (3.2), we know that $\Psi'(\sigma) = 1/\lambda(\sigma) > 0$, so there exists a function $g : [0, m] \to [0, 1]$, with $g(0) = 0$ such that

$$g(\tau) \equiv \sigma(0, \tau/c_o) \text{ is the inverse of the function } \Psi(\sigma). \tag{3.5}$$

Setting $X = 1$ in (2.10), using (3.4), (3.5) and (3.1) we obtain the following functional equation for $g(\psi)$:

$$g(C(T)) = \frac{c(1, T)}{c_o} g(c_o T) = \frac{C'(T)}{c_o} g(c_o T) \quad \text{for} \quad T \geqslant 0. \tag{3.6}$$

Finally, denoting $\tau \equiv c_o T$ so that

$$\frac{dT}{d\tau} = \frac{1}{c_o}, \tag{3.7a}$$

$$B = c_o A \tag{3.7b}$$

$$D(\tau) \equiv C(\tau/c_o), \tag{3.7c}$$

equation (3.6) can be rewritten as

$$g(D(\tau)) = D'(\tau) g(\tau) \quad \text{for} \quad \tau \in [0, B], \tag{3.8}$$

which is known as Julia's equation in g for the prescribed D, see [14].

If the injected concentration $c(0, T)$ is not constant, then a functional equation to (3.8) can be obtained (see details in [2]). All the results obtained here are still valid for this functional equation, but for simplicity we take $c(0, T)$ as a constant.

3.1. Recovery of the filtration function

In this section, we show how to recover the filtration function $\lambda(\sigma)$ using (3.8), based on experimental histories of effluent inject concentrations. The effluent concentration data (T_j, c_e^j), $j = 1 \ldots, J$ is measured in the laboratory; it is assumed to be positive for positive times. The injected concentration c_o is known provided particles are not retained at the injection well (this occurs if there is no cake formation [1]. Ideally, the recovered $\lambda(\sigma)$ should yield a concentration such that $c(1, T_j) = c_e^j$, $j = 1, \ldots, N$. As in (3.1) and (3.7c) for the direct problem we redefine

$$D(\tau) = \int_0^{\tau/c_o} c_e(s)\,\mathrm{d}s, \qquad D : [0, B], \to \mathbb{R}, \tag{3.9}$$

where $c_e(T)$, with $T \in [0, A]$ is an approximation of the experimental data histories (T_j, c_e^j), $j = 1 \ldots, J$. This approximation will be obtained by preprocessing, which is explained later. Motivated by the fact that the filtration function λ should be positive and because of remark (2.4) we make the following.

Assumption 3.1. *The function $c_e(T)$ is $C^2[0, A]$ and $c_e(T) < c_o$ for all T.*

The existence and uniqueness of the solution of (3.8) is studied in [2]. The result is

Theorem 3.2. *Consider the Banach space*

$$G_2^0 = \{g \in C^2[0, B], \quad g(0) = 0\} \tag{3.10}$$

with norm

$$\|g\| = ||g||_\infty + ||g'||_\infty + ||g''||_\infty. \tag{3.11}$$

Let $D : [0, B] \to \mathbb{R}$ given by (3.9) be a C^2 monotone increasing function, such that $D(0) = 0$, $0 < D(\tau) < \tau$ and $D'(0) < 1$. Then the functional equation (3.8) has a solution $g \in G_2^0$, which is uniquely defined by the value of the derivative $g'(0)$.

As shown in [2], the solution of (3.8) is

$$g(\tau) = g'(0) \lim_{n\to\infty} \frac{\tau_n}{\prod_{k=0}^{n-1} D'(\tau_k)} = g'(0) \prod_{n=0}^{\infty} \frac{D(\tau_n)}{D'(\tau_n)\tau_n}; \tag{3.12}$$

$$\tau_n = D^n(\tau), \tag{3.13}$$

where D is given in (3.9), $\tau \in [0, B]$ is arbitrary and $\tau_n = D(D^{n-1}(\tau))$ with $D^0(\tau) = \tau$.

As also shown in [2], from (2.4), (2.9*b*), (3.2) and (3.5) we have

$$g'(0) = \lambda(0) = -\log(c_e(0)/c_o). \tag{3.14}$$

Remark 3.3. From (3.4*a*), we obtain that $\sigma(0, T) = g(c_o T)$ and $\Psi(\sigma(0, T)) = c_o T$ so that

$$\Psi(g(c_o T)) = c_o T. \tag{3.15}$$

Now from (3.2), (3.5), (3.6) and (3.7) we obtain

$$\lambda(\sigma(0, T)) = \frac{1}{\Psi'(\sigma(0, T))} = \frac{1}{c_o} \frac{\mathrm{d}g}{\mathrm{d}T}(g^{-1}(\sigma(0, T))) = \frac{\mathrm{d}g}{\mathrm{d}\tau}(\tau) = g'(\tau), \tag{3.16}$$

where we again used $\tau \equiv c_o T$.

Data c_o, $c_e(T)$ that satisfy assumption 3.1 are said to *conform* to filtration theory if there is a $\lambda(\sigma) > 0$ in $0 < \sigma < 1$ for which the solution of (2.1)–(2.4) gives rise to a function $c(X, T)$ such that $c(1, T) = c_e(T)$ for $T \in [0, A]$. These are essential assumptions to obtain a consistent model and physically adequate results. In order to obtain a robust inversion method, we require that the approximation $c_e(T)$ of the data is an analytic function (see sections 5 and 7). In such a case, the filtration function is also analytic. So, we will require analyticity from section 5 on.

4. Monotonicity

In this work, we focus on the case where the filtration function is a monotone decreasing function of the particle concentration. This is true in many applications, according to the extensive review provided in Hertzig *et al* [10], including an example by Ives [11]. This assumption is false in other applications as, for example, the work of Ives [12, 13]). See also more recent cases presented in [5, 6, 22]. The general case, where the filtration function may be non-monotone, lies outside the scope of this paper.

An inversion method can be developed for non-monotone filtration function by means of optimization techniques as in [17]. Generalizing the functional equation method for such a case remains a challenge.

In this section, we present conditions for the monotonicity of the solution of equation (3.8) and its derivative. Here we assume that $D' > 0$, which means that effluent concentration of particles is always positive. Differentiating (3.8), dividing by D' and denoting

$$h(\tau) = \frac{D''}{D'}(\tau)g(\tau) \tag{4.1}$$

yields

$$g'(D(\tau)) = g'(\tau) + h(\tau) \tag{4.2}$$

for $\tau \geqslant 0$. If the function h in (4.2) is monotone increasing, then one obtains sufficient conditions for the filtration function to be monotone decreasing. We will need the following

Assumption 4.1. *We assume that* $\frac{D''}{D'}(\tau) = (\log(c_e(\tau/c_o)))'$ *is a monotone increasing function in* $[0, A]$.

Assumption 4.2. For $\tau \in [0, B]$, we assume that

$$(D'(\tau))^2\tau - D(\tau)(D'(\tau)\tau)' > 0. \tag{4.3}$$

Remark 4.3. Note that $(D''/D')' = (D'''D' - (D'')^2)/(D')^2$ and recall that $D' = c_e/c_o$. Thus, if $c_e''c_e - (c_e')^2 > 0$ then assumption 4.1 holds.

Next, we establish the monotonicity of the solution g of the functional equation (3.8), which we use to prove the monotonicity of the function h in (4.1).

Lemma 4.4. *Assume that the hypotheses of theorem 3.2 are satisfied and let D in (3.9) satisfy assumption 4.2 for all* $\tau \in [0, B]$. *Then the solution g of the functional equation (3.8) is monotone increasing.*

Proof. We define

$$G(\tau) = D(\tau)/D'(\tau)\tau \tag{4.4}$$

and take $\tau < \xi$. Let g be the solution of (3.8). Note that from (3.12) we have

$$g(\tau)/g(\xi) = \prod_{j=1}^{\infty}(G(\tau_j)/G(\xi_j)), \tag{4.5}$$

where $\tau_j = D^j(\tau)$ and $\xi_j = D^h(\xi)$ as in (3.13). Note G' is the function with numerator that coincides with the LHS of inequality (4.3) and denominator $(D(\tau)'\tau)^2$. Then assumption 4.2 ensures that $G' > 0$ for all $\tau \in [0, B]$.

Thus, using the facts that $\tau_j < \xi_j$, from remark 4.8 and that the function G in (4.4) is increasing, we have $G(\tau_j) < G(\xi_j)$ for $j = 1, 2\ldots$ and as a consequence of (3.13) we have $g(\tau) < g(\xi)$, so g is increasing. □

The main result of this section is the following.

Theorem 4.5. *Assume that (i) assumption 4.1 is satisfied and that (ii) the solution g of (3.8) is a monotone increasing function. Then (iii) the filtration function is monotone decreasing. Assumption 4.2 is a sufficient condition for (ii), so that if (i) is valid then (iii) holds.*

In order to prove theorem 4.5, we use the following lemma, which is a direct application of theorem 2.3.6, page 65 of [14] to our case.

Lemma 4.6. *Let g in G_2^0 satisfy (3.8). If the function h in (4.1) is monotone increasing and $\lim_{\tau\to 0} h(\tau) = 0$, then equation (4.2) has a unique one-parameter family of monotone decreasing solutions $g'(\tau)$. Moreover the solution satisfies*

$$g'(\tau) = g'(\xi) - \sum_{n=0}^{\infty}(h(\tau_n) - h(\xi_n)). \tag{4.6}$$

Here, ξ is an arbitrary value in $[0, B]$; $\tau_n = D^n(\tau)$, $\xi_n = D^n(\xi)$ as in (3.13)

Remark 4.7. Assuming uniform convergence for the series in ξ given by $\sum_{n=0}^{\infty} h(\xi_n)$, where $\xi_n = D^n(\xi)$ (this is the case in remark 6.6), we can take $\xi \to 0$ in (4.6) and using $g(0) = 0$, formula (4.6) can be rewritten as

$$g'(\tau) = g'(0) - \sum_{n=0}^{\infty} h(\tau_n). \tag{4.7}$$

Using equation (3.8), we obtain $g(\tau_n) = \prod_{j=1}^{n-1} D'(\tau_{n-j})g(\tau)$. If we define

$$Q(\tau) = \frac{D''(\tau)}{D'(\tau)} + \frac{D''(\tau_1)}{D'(\tau_1)}D'(\tau) + \sum_{n=2}^{\infty}\frac{D''(\tau_n)}{D'(\tau_n)}\prod_{j=1}^{n-1}D'(\tau_{n-j}), \tag{4.8}$$

then equation (4.7) becomes

$$g'(\tau) = g'(0) - Q(\tau)g(\tau). \tag{4.9}$$

Equation (4.9) is an explicit formula for $\lambda(\sigma)$, since $\lambda(\sigma) = g'(\tau)$. The quantity $\lambda(0)$ is given in terms of experimental data in (3.14), $Q(\tau)$ in (4.8) and g in (3.12).

Remark 4.8. If the function h in (4.1) is monotone increasing, then λ is monotone decreasing. To see this fact, we make $\tau < \xi$ in (4.6) and use $\tau_n < \xi_n$, as in (3.13), which follows from the fact that D^n is an increasing function. Then from equation (4.6), we obtain that $\lambda = g'$ is monotone decreasing.

We can now prove theorem 4.5. The solution $g \in G_2^0$ of equation (3.8) given in (3.12) is monotone increasing by lemma 4.4 and $D''/D'(\tau) = (\log(c_e(\tau/c_o)))'$ is monotone increasing by assumption 4.1, so h is monotone increasing as well. Finally, from lemma 4.6 and remark 4.8, we obtain that the solution g' of the functional equation (4.2) is monotone decreasing. Recalling that $\lambda = g'(\tau)$, we obtain that the filtration function is monotone decreasing.

5. Analyticity

We now prove that $\lambda(\sigma)$ is a real analytic function provided the effluent concentration $c_e(T)$ is analytic. We use the definition that the function f on closed bounded interval is real analytic if there is a complex analytic extension of f to an open set $G \subset \mathbb{C}$ which contains the interval.

Lemma 5.1. *Under the assumptions of theorem 3.2, if c_e is real analytic on* $[0, A]$*, then the solution of (3.8) is real analytic.*

To prove analyticity, we consider, for simplicity, that (3.8) has a complex solution in the real line. To do so, we use the complex extension of c_e to an open set in $\mathbb{C}$ containing $[0, A]$. We now obtain a complex analytical solution of (3.8, by applying theorem 4.2.1 page 151 of [14]; restricting it to the real line, we obtain the real analytic solution.

6. Stability

Conditions for numerical stability of a computer implementation of a recovery method for λ given c_e and c_o were presented in [2]. However, stability was not established from intrinsic properties of the functional equation (3.8). We fill this gap in this section under assumptions 6.1 and 6.2, in addition to the hypotheses on $D(\tau)$ of theorem 3.2: we will need the following.

Assumption 6.1. *We assume that c_o, $c_e(T)$ are such that $D(\tau)$ defined in (3.9) is a non-negative C^3 function for $0 \leqslant \tau \leqslant B$ with B defined in (3.7a) satisfying*

$$0 \leqslant D(\tau) < \tau, \ 0 < D'(\tau) < d < 1 \ \text{ for } \ 0 \leqslant \tau \leqslant B; \tag{6.1a}$$

$$D(0) = 0 \ \text{ and } \ \mathrm{D}''(0) \neq 0. \tag{6.1b}$$

Assumption 6.2. *We impose certain new restrictions on the data. The effluent particle concentration $c_e(T)$ is restricted to*

$$\mathcal{M} = \{c_e \in C^2[0, A] : 0 < r_1 < c_e(T) \leqslant r_2, \ 0 \leqslant r_3 \leqslant c_e'(T) < r_4; r_5 \leqslant c_e''(T) < r_6\},$$

for certain constants $r_1, \ldots, r_6$. Moreover we take the inlet concentration of particles restricted as follows:

$$0 < r_7 < c_o < r_8, \tag{6.2}$$

for certain constants r_7, r_8.

Remark 6.3. Note that bounds for the constant injected particles concentration c_o in (6.2) are used. This inequality means either that experimental errors in the data series can be included in the model or that several experiments can be performed with different c_o. More precisely, we will need that all the possible input data c_o be uniformly bounded by the constants r_7, r_8.

We assume that there exists a constant d_1 such that

$$(D''(\tau) - 2D'(\tau))/D'(0) < d_1, \quad \text{for all} \quad \tau \in [0, B], \tag{6.3}$$

which can be obtained from the properties on D in assumptions 6.1 and 6.2.

Remark 6.4. In the case of constant filtration function, the solution (2.11) satisfies assumptions 6.1 and 6.2 for $X \in [0, 1]$ and $T \in [0, A]$.

Remark 6.5. From (3.14) and assumptions 6.1 and 6.2, we can see that $g'(0) \neq 0$, and $g'(0) < d_2$ for a certain constant d_2. These conditions are imposed on the class of solutions $\mathcal{G}$ of the functional equation (3.8).

Remark 6.6. It is possible to verify that, under assumption 6.1, the series $\sum_{n=0}^{\infty} h(\xi_n)$, where $\xi_n = D^n(\xi)$ in (4.6), converges uniformly for all $\xi \in [0, B]$.

The main result of this section is the following stability result.

Theorem 6.7. *Let consider c_{o1}, c_{o2}, $c_{e1}(T)$, $c_{e2}(T)$ and define*

$$D_1(\tau) \equiv \int_0^{\tau/c_{o1}} c_{e1}(s)\,\mathrm{d}s, \quad D_2(\tau) \equiv \int_0^{\tau/c_{o2}} c_{e2}(s)\,\mathrm{d}s, \tag{6.4}$$

such that assumptions 6.1 and 6.2 are satisfied. Let us denote by $\lambda_1 = g_1'$ and $\lambda_2 = g_2'$, where g_1, g_2 are the solutions of the functional equation (3.8) associated with D_1 and D_2 respectively. Then there exist constants m_1, m_2 independent of $c_{e1}, c_{e2}, c_{o1}, c_{o2}$ such that

$$||\lambda_1 - \lambda_2||_\infty \leqslant m_1|c_{o1} - c_{o2}| + m_2(||c_{e1} - c_{e2}||_\infty + ||c_{e1}' - c_{e2}'||_\infty). \tag{6.5}$$

In (6.5), we take the supremum norm on $[0, 1]$ for the filtration function and on $[0, A]$ for the effluent concentration function and its derivative. The proof of theorem 6.7 is built in terms of a number of lemmas. The idea is to estimate how changes in the concentration c_o and c_e modify D and its derivatives (lemma 6.8, theorem 6.9, lemma 6.11). Then, we estimate how changes in D affect $g(\tau)$ and $Q(\tau)$ in (4.9); these estimates are obtained in lemmas 6.12–6.18.

Accurate estimates for the constants appearing in equation (6.5) can be easily collected from the lemmas. They are useful to evaluate how sensitive the filtration function is to the concentration data.

First, we prove that the solution of equation (4.2) depends continuously on the particle concentration c_o, $c_e(T)$, under assumptions 6.1 and 6.2.

Lemma 6.8. *Let D_1 correspond with c_{o1}, c_{e1} and D_2 correspond with c_{o2}, c_{e2}, as in (6.4). Under the assumptions 6.1 and 6.2, and denoting $'$ as $\mathrm{d}/\mathrm{d}\tau$, there exist constants $N_1, \ldots, N_8$ that depend only on $r_1, \ldots, r_8$ such that*

(i) $||D_1'' - D_2''||_\infty$ *is bounded by* $N_1|c_{o1} - c_{o2}| + N_2||c_{e1}' - c_{e2}'||_\infty$,
(ii) $||D_1' - D_2'||_\infty$ *is bounded by* $N_3|c_{o1} - c_{o2}| + N_4||c_{e1} - c_{e2}||_\infty$,
(iii) $||D_1 - D_2||_\infty$ *is bounded by* $N_5|c_{o1} - c_{o2}| + N_6||c_{e1} - c_{e2}||_\infty$.

Proof. The derivatives of D_1 and D_2 defined in (6.4) are $D_1'(\tau) = c_{e1}(\tau/c_{o1})/c_{o1}$, $D_1''(\tau) = c_{e1}'(\tau/c_{o1})/(c_{o1})^2$, etc. We have

$$|c_{e1}'/(c_{o1})^2 - c_{e2}'/(c_{o2})^2| \leqslant (c_{o1}c_{o2})^{-2}((c_{o2})^2|c_{e1}' - c_{e2}'| + |c_{e2}'||(c_{o1})^2 - (c_{o2})^2|). \tag{6.6}$$

Taking $N_1 = 2r_4r_8/r_7^4$ and $N_2 = r_8^2/r_7^4$, where r_4, r_7, r_8 are defined in assumptions 6.1 and 6.2, we obtain

$$||D_1'' - D_2''||_\infty \leqslant N_1|c_{o1} - c_{o2}| + N_2||c_{e1}' - c_{e2}'||_\infty.$$

We obtain analogously the inequalities (ii) and (iii) . □

In [2] and [1], the following theorem was proved for d in (6.1*b*), d_1 in (6.3) and d_2 in remark 6.5.

Theorem 6.9. *Under the assumptions 6.1 and 6.2, any solution g of the functional equation (3.8) is uniformly bounded by $d_2\mathrm{e}^{d_3/(1-d)}$ with a constant d_3, and its derivative g' is uniformly bounded by $d_2\mathrm{e}^{d_1/2(1-d)}$.*

Remark 6.10. The bound for the derivative g' in the current paper is correct; it is different from the bound appearing in [2], where a typographical error was made.

Lemma 6.11. *Under the hypotheses of theorem 6.7 and $D_1(B) < D_2(B)$ (without loss of generality), there exists a constant M independent of $c_{e1}, c_{e2}, c_{o1}, c_{o2}$ such that the following inequalities hold:*

(i) $||D_1^{-1} - D_2^{-1}||_\infty$ *is bounded by* $M||D_1 - D_2||_\infty$,
(ii) $|D_1'(D_1^{-1}(s)) - D_2'(D_2^{-1}(s))|$ *is bounded by*
$||D_1' - D_2'||_\infty + ||D_2''||_\infty M||D_1 - D_2||_\infty$, *for all* $s \in [0, D_1(B)]$.

Proof. (i) Taking $s_1, s_2 \in D_1[0, B] \subset D_2[0, B]$ we have

$$D_1^{-1}(s_1) - D_1^{-1}(s_2) = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}\alpha}\left(D_1^{-1}(\alpha s_1 + (1-\alpha)s_2)\right) \mathrm{d}\alpha. \tag{6.7}$$

Since $D_1 \in \mathcal{M}$, then if $s = D(\tau)$, $(D_1^{-1}(s))' = 1/D_1'(\tau) < (c_{o1}/r_1)$, so we obtain

$$\left|D_1^{-1}(s_1) - D_1^{-1}(s_2)\right| \leqslant (c_{o1}/r_1)|s_1 - s_2|. \tag{6.8}$$

For a fixed s_1, let $s_2 = D_1(D_2^{-1}(s_1))$; it follows that $D_1^{-1}(s_2) = D_2^{-1}(s_1)$. From (6.2) and (6.8), we have

$$\begin{aligned}\left|D_1^{-1}(s_1) - D_2^{-1}(s_1)\right| &= \left|D_1^{-1}(s_1) - D_1^{-1}(s_2)\right| \leqslant (c_{o1}/r_1)|s_1 - s_2| \leqslant (r_8/r_1)\big|D_1\big(D_2^{-1}(s_1)\big)\\ &\quad - D_2\big(D_2^{-1}(s_1)\big)\big|.\end{aligned} \tag{6.9}$$

From (6.9) we see that (i) holds. To prove (ii), note that for $\Upsilon_1 = D_1^{-1}(s)$ and $\Upsilon_2 = D_2^{-1}(s)$

$$|D_1'(\Upsilon_1) - D_2'(\Upsilon_2)| \leqslant |D_1'(\Upsilon_1) - D_2'(\Upsilon_1)| + |D_2'(\Upsilon_2) - D_2'(\Upsilon_1)| \tag{6.10}$$

From (6.10) and using the mean value theorem, we obtain

$$|D_1'\big(D_1^{-1}(s)\big) - D_2'\big(D_2^{-1}(s)\big)| \leqslant ||D_1' - D_2'||_\infty + ||D_2''||_\infty\left|D_1^{-1}(s) - D_2^{-1}(s)\right|; \tag{6.11}$$

using (i) in (6.11) we obtain (ii) . □

Now, we verify the validity of the following lemma.

Lemma 6.12. *Under the hypotheses of theorem 6.7, let us denote by g_1 and g_2 the solutions of equation (3.8) in $\mathcal{G}$ satisfying (3.14) with corresponding D_1 and D_2 defined on* $[0, B]$ *satisfying the condition (6.1a). Then there exist constants υ_1, υ_2 independent of $c_{e1}, c_{e2}, c_{o1}, c_{o2}$, such that*

$$||g_1 - g_2||_\infty \leqslant \upsilon_1||c_{e1} - c_{e2}||_\infty + \upsilon_2|c_{o1} - c_{o2}|. \tag{6.12}$$

Proof. Taking $s = D(\tau)$, equation (3.8) can be rewritten as

$$g(s) = D'(D^{-1}(s))g(D^{-1}(s)). \tag{6.13}$$

Now, using (6.13) and the notation $\Upsilon_1 = D_1^{-1}(s)$ and $\Upsilon_2 = D_2^{-1}(s)$ we obtain

$$|g_1(s) - g_2(s)| = |D_1'(\Upsilon_1)g_1(\Upsilon_1) - D_2'(\Upsilon_2)g_2(\Upsilon_2)|.$$

Note that

$$|g_1(s) - g_2(s)| \leqslant |D_1'(\Upsilon_1)g_1(\Upsilon_1) - D_2'(\Upsilon_2)g_2(\Upsilon_1)| + |D_2'(\Upsilon_2)(g_2(\Upsilon_2) - g_2(\Upsilon_1))| \tag{6.14}$$

Using the mean value theorem and the definition of Υ_1 and Υ_2, we have

$$|D_2'(\Upsilon_2)(g_2(\Upsilon_2) - g_2(\Upsilon_1))| \leqslant ||D_2'||_\infty||g_2'||_\infty\left|\left|D_2^{-1} - D_1^{-1}\right|\right|_\infty. \tag{6.15}$$

Moreover

$$\begin{aligned}\left|D_1'(\Upsilon_1)g_1(\Upsilon_1) - D_2'(\Upsilon_2)g_2(\Upsilon_1)\right| &\leqslant |g_1(\Upsilon_1)||D_1'(\Upsilon_1) - D_2'(\Upsilon_2)|\\ &\quad + \left|D_2'(\Upsilon_2)\right|\left|g_2(\Upsilon_1) - g_1(\Upsilon_1)\right|\end{aligned} \tag{6.16}$$

or

$$|D_1'(\Upsilon_1)g_1(\Upsilon_1) - D_2'(\Upsilon_2)g_2(\Upsilon_1)| \leqslant ||g_1||_\infty|D_1'(\Upsilon_1) - D_2'(\Upsilon_2)| + ||D_2'||_\infty|(g_2 - g_1)(\Upsilon_1)| \tag{6.17}$$

Since $||D_2'||_\infty < d < 1$, we have from (6.14), (6.15) and (6.17) that

$$(1 - d)||g_1 - g_2||_\infty \leqslant ||g_2'||_\infty|||D_2'||_\infty||D_2^{-1} - D_1^{-1}||_\infty + ||g_1||_\infty|D_1'(\Upsilon_1) - D_2'(\Upsilon_2)| \tag{6.18}$$

Finally, using theorem 6.9, lemmas 6.8, 6.11 and equations (6.10), (6.11) we obtain (6.12). □

Another useful inequality is given in the following lemma.

Lemma 6.13. *Let $\tau_{m,i} = D_i^m(\tau)$, (see equation (3.13)) with i = 1, 2 and* $m = 0, 1, \ldots$. *The following inequalities hold (with d given in assumption* 6.1*b):*

$$|\tau_{m,2} - \tau_{m,1}| \leqslant ||D_1 - D_2||_\infty \sum_{k=0}^{m} d^k \tag{6.19}$$

Proof. We prove (6.19) by induction. It is easy to see that the inequality (6.19) is valid for $m = 1$ using $\tau_{1,1} = D_1(\tau)$ and $\tau_{1,2} = D_2(\tau)$. Now we assume that (6.19) is valid for $m - 1$, i.e.

$$|\tau_{m-1,2} - \tau_{m-1,1}| \leqslant \sum_{k=0}^{m-1} d^k ||D_1 - D_2||_\infty. \tag{6.20}$$

Note that

$$|\tau_{m,2} - \tau_{m,1}| \leqslant |D_2\big(D_2^{m-1}(\tau)\big) - D_2\big(D_1^{m-1}(\tau)\big)| + |D_1\big(D_1^{m-1}(\tau)\big) - D_2\big(D_1^{m-1}(\tau)\big)|. \tag{6.21}$$

From the mean value theorem and (6.1), (6.21), we have

$$\begin{aligned} |\tau_{m,2} - \tau_{m,1}| &\leqslant ||D_2'||_\infty |\tau_{m-1,2} - \tau_{m-1,1}| + ||D_1 - D_2||_\infty \\ &\leqslant d|\tau_{m-1,2} - \tau_{m-1,1}| + ||D_1 - D_2||_\infty. \end{aligned} \tag{6.22}$$

Using (6.20) in (6.22), we obtain

$$|\tau_{m,2} - \tau_{m,1}| \leqslant d||D_1 - D_2||_\infty \sum_{k=0}^{m-1} d^k + ||D_1 - D_2||_\infty. \tag{6.23}$$

or

$$|\tau_{m,2} - \tau_{m,1}| \leqslant ||D_1 - D_2||_\infty \left(1 + d\sum_{k=0}^{m-1} d^k\right). \tag{6.24}$$

Finally, from (6.24) we obtain that (6.19) holds. □

Lemma 6.14. *Let $\tau_{m,i} = D_i^m(\tau)$, $m = 0, 1, 2, \ldots$ (see equation (3.13)) with i=1,2 and m=0,1,.... The following inequalities hold.*

(i) $|D_1''(\tau_{m,1}) - D_2''(\tau_{m,2})| \leqslant ||D_1'' - D_2''||_\infty + ||D_2'''||_\infty ||D_1 - D_2||_\infty \sum_{k=0}^{m} d^k.$

(ii) $|D_1'(\tau_{m,1}) - D_2'(\tau_{m,2})| \leqslant ||D_1' - D_2'||_\infty + ||D_2''||_\infty ||D_1 - D_2||_\infty \sum_{k=0}^{m} d^k.$

Proof. Note that

$$|D_1''(\tau_{m,1}) - D_2''(\tau_{m,2})| \leqslant |D_1''(\tau_{m,1}) - D_2''(\tau_{m,1})| + |D_2''(\tau_{m,1}) - D_2''(\tau_{m,2})|. \tag{6.25}$$

Now using the mean value theorem, we obtain

$$|D_1''(\tau_{m,1}) - D_2''(\tau_{m,2})| \leqslant ||D_1'' - D_2''||_\infty + ||D_2'''||_\infty |\tau_{m,1} - \tau_{m,2}|, \tag{6.26}$$

Finally using lemma 6.13 in (6.26) we obtain (*i*). The inequality (*ii*) is proved analogously. □

Moreover we will use the following.

Lemma 6.15. *Let $\tau_{m,i}$ be as in lemma 6.13 and* $m = 1, \ldots$. *The following inequalities hold:*

$$\sum_{j=1}^{m} |D_1'(\tau_{m-j,1}) - D_2'(\tau_{m-j,2})| \leqslant ||D_2''||_\infty \left(\sum_{j=1}^{m}\sum_{k=0}^{m-j} d^k\right) ||D_1 - D_2||_\infty + m||D_1' - D_2'||_\infty. \tag{6.27}$$

Proof. We have that

$$|D_1'(\tau_{m-j,1}) - D_2'(\tau_{m-j,2})| \leqslant |D_1'(\tau_{m-j,1}) - D_2'(\tau_{m-j,1})| + |D_2'(\tau_{m-j,2}) - D_2'(\tau_{m-j,1})|. \quad (6.28)$$

Therefore, by the mean value theorem

$$|D_1'(\tau_{m-j,1}) - D_2'(\tau_{m-j,2})| \leqslant ||D_1' - D_2'||_\infty + ||D_2''||_\infty |\tau_{m-j,2} - \tau_{m-j,1}|. \quad (6.29)$$

Adding (6.29) from $j = 1$ to m and using lemma 6.13 we obtain that (6.27) holds. □

Lemma 6.16. *Let $\tau_{m,i}$ be as in lemma 6.13 and $m = 1, \ldots$. The following inequalities hold:*

$$0 < \tau_{m,i}' = \prod_{j=1}^{m} D_i'(\tau_{m-j,i}) \leqslant d^m, \quad i = 1, 2. \quad (6.30)$$

Proof. Inequalities (6.30) are an immediate application of (6.1*b*) i.e. $0 < D_i'(\tau) < d$. □

Lemma 6.16 immediately gives

$$|(\eta\tau_{m,1} + (1-\eta)\tau_{m,2})'| \leqslant d^m \quad \text{for } m = 1, \ldots. \quad (6.31)$$

Let us denote

$$Q_i(\tau) = \sum_{n=0}^{\infty} \frac{D_i''(\tau_{n,i})}{D_i'(\tau_{n,i})} \prod_{j=1}^{n} D_i'(\tau_{n-j,i}) \quad \text{or} \quad Q_i(\tau) = \sum_{n=0}^{\infty} \frac{D_i''(\tau_{n,i})}{D_i'(\tau_{n,i})} \tau_{n,i}', \; i = 1, 2.$$

Note that (6.31) leads to

$$|Q_i(\tau)| \leqslant \left\| \frac{D_i''}{D_i'} \right\|_\infty \sum_{n=0}^{\infty} d^n = \left\| \frac{D_i''}{D_i'} \right\|_\infty \frac{1}{1-d} \leqslant K, \quad i = 1, 2.$$

Note that $D_i'' = c_{e,i}'/(c_{o,i})^2$ and $D_i' = c_{e,i}/c_{o,i}$. So since $c_{e,i}$ belong to $\mathcal{M}$ (see assumptions 6.2) and $c_{o,i}$ satisfy (6.2) then D_i''/D_i' with $i = 1, 2$ are uniformly bounded.

Lemma 6.17. *For $m = 1, 2, \ldots$ the following inequalities hold:*

$$\left| \prod_{l=1}^{m} D_1'(\tau_{l,1}) - \prod_{l=1}^{m} D_2'(\tau_{l,2}) \right| \leqslant d^{m-1} \sum_{j=1}^{m} |D_1'(\tau_{j,1}) - D_2'(\tau_{j,2})|, \quad (6.32)$$

Proof. We prove (6.32) by induction. For $m = 2$, we have

$$\begin{aligned} \left| \prod_{l=1}^{2} D_1'(\tau_{l,1}) - \prod_{l=1}^{2} D_2'(\tau_{l,2}) \right| &= |D_1'(\tau_{1,1})D_1'(\tau_{2,1}) - D_2'(\tau_{1,2})D_2'(\tau_{2,2})| \\ &\leqslant |D_1'(\tau_{1,1})D_1'(\tau_{2,1}) - D_2'(\tau_{1,2})D_1'(\tau_{2,1})| \\ &\quad + |D_2'(\tau_{1,2})D_1'(\tau_{2,1}) - D_2'(\tau_{1,2})D_2'(\tau_{2,2})|. \end{aligned} \quad (6.33)$$

It follows that

$$\begin{aligned} \left| \prod_{l=1}^{2} D_1'(\tau_{l,1}) - \prod_{l=1}^{2} D_2'(\tau_{l,2}) \right| &\leqslant |D_1'(\tau_{2,1})||D_1'(\tau_{1,1}) - D_2'(\tau_{1,2})| \\ &\quad + |D_2'(\tau_{1,2})||D_1'(\tau_{2,1}) - D_2'(\tau_{2,2})| \\ &\leqslant d(|D_1'(\tau_{1,1}) - D_2'(\tau_{1,2})| + |D_1'(\tau_{2,1}) - D_2'(\tau_{2,2})|). \end{aligned} \quad (6.34)$$

From (6.34) we see that (6.32) holds for $m = 2$. Now we assume that (6.32) is valid for $m - 1$, i.e.

$$\left| \prod_{l=1}^{m-1} D_1'(\tau_{l,1}) - \prod_{l=1}^{m-1} D_2'(\tau_{l,2}) \right| \leqslant d^{m-2} \sum_{j=1}^{m-1} |D_1'(\tau_{j,1}) - D_2'(\tau_{j,2})|. \quad (6.35)$$

Finally, we prove that (6.32) is valid for m. Note that

$$\left|\prod_{l=1}^{m} D_1'(\tau_{l,1}) - \prod_{l=1}^{m} D_2'(\tau_{l,2})\right| \leqslant |D_1'(\tau_{m,1})| \left|\prod_{l=1}^{m-1} D_1'(\tau_{l,1}) - \prod_{l=1}^{m-1} D_2'(\tau_{l,2})\right| + \left(\prod_{l=1}^{m-1} |D_2'(\tau_l^2)|\right) |D_1'(\tau_{m,1}) - D_2'(\tau_{m,2})|. \tag{6.36}$$

Substituting (6.35) in (6.36), we obtain

$$\left|\prod_{l=1}^{m} D_1'(\tau_{l,1}) - \prod_{l=1}^{m} D_2'(\tau_{l,2})\right| \leqslant |D_1'(\tau_{m,1})| d^{m-2} \sum_{j=1}^{m-1} |D_1'(\tau_{j,1}) - D_2'(\tau_{j,2})| + \left(\prod_{l=1}^{m-1} |D_2'(\tau_{l,2})|\right) |D_1'(\tau_{m,1}) - D_2'(\tau_{m,2})|. \tag{6.37}$$

Using (6.1*b*) and (6.30) in (6.37), we obtain (6.32). □

Lemma 6.18. *Under the hypotheses of theorem 6.7, let us denote by g_1, g_2 the solutions of equation (3.8) in $\mathcal{G}$ satisfying (3.14) and corresponding to D_1 and D_2. Then there exist certain constants v_1, v_2, v_3 independent of c_{e1}, c_{e2}, c_{o1}, c_{o2} such that*

$$|Q_1(\tau)g_1(\tau) - Q_2(\tau)g_2(\tau)| \leqslant v_1 ||c_{e1} - c_{e2}||_\infty + v_2 ||c_{e1}' - c_{e2}'||_\infty + v_3 |c_{o1} - c_{o2}|. \tag{6.38}$$

Proof. Let us define

$$G_n(\tau, \alpha) = \alpha D_1''(\tau_{n,1}) + (1-\alpha) D_2''(\tau_{n,2}),\ F_n(\tau, \alpha) = \alpha D_1'(\tau_{n,1}) + (1-\alpha) D_2'(\tau_{n,2}).$$

Using the convention that $(\alpha\tau_{0,1} + (1-\alpha)\tau_{0,2})' = 1$, we define (see (4.8))

$$Q(\tau, \alpha) = \sum_{n=0}^{\infty} \frac{G_n(\tau, \alpha)}{F_n(\tau, \alpha)} (\alpha\tau_{n,1} + (1-\alpha)\tau_{n,2})', \tag{6.39}$$

where $\alpha \in [0, 1]$. Note that $Q(\tau, 1) = Q_1(\tau)$ and $Q(\tau, 0) = Q_2(\tau)$. Now

$$|Q_1(\tau)g_1(\tau) - Q_2(\tau)g_2(\tau)| \leqslant |Q_1(\tau)g_1(\tau) - Q_1(\tau)g_2(\tau)| + |Q_2(\tau)g_2(\tau) - Q_1(\tau)g_2(\tau)|,$$

or

$$|Q_1(\tau)g_1(\tau) - Q_2(\tau)g_2(\tau)| \leqslant |Q_1(\tau)||g_1(\tau) - g_2(\tau)| + |g_2(\tau)||Q_2(\tau) - Q_1(\tau)|.$$

Thus, it is necessary to bound $|Q_2(\tau) - Q_1(\tau)|$. From the mean value theorem there exists $\eta \in (0, 1)$ such that

$$Q_2(\tau) - Q_1(\tau) = \frac{\partial Q(\tau, \eta)}{\partial \alpha}. \tag{6.40}$$

From (6.39) and (6.40), we have that for $Q(\tau, \eta)$

$$\frac{\partial Q}{\partial \alpha} = \sum_{n=0}^{\infty} \left(\frac{(D_1''(\tau_{n,1}) - D_2''(\tau_{n,2}))F_n - (D_1'(\tau_{n,1}) - D_2'(\tau_{n,2}))G_n}{(F_n)^2} \right) (\eta\tau_{n,1} + (1-\eta)\tau_{n,2})' + \sum_{n=0}^{\infty} (G_n/F_n)(\tau_{n,1} - \tau_{n,2})', \tag{6.41}$$

where $(\tau_{0,1} - \tau_{0,2})' = 0$ and $F_n = F_n(\tau, \alpha)$ and $G_n = G_n(\tau, \alpha)$.

Using lemma 6.17 for $(\tau_{n,1}-\tau_{n,2})' = \prod_{j=1}^{n} D_1'(\tau_{n-j,1}) - \prod_{j=1}^{n} D_2'(\tau_{n-j,2})$, the following inequality holds:

$$|(\tau_{n,1}-\tau_{n,2})'| \leqslant d^{n-1}\sum_{j=1}^{n}|D_1'(\tau_{n-j,1}) - D_2'(\tau_{n-j,2})|. \tag{6.42}$$

Moreover, using lemma 6.15 we obtain the following inequality:

$$|(\tau_{n,1}-\tau_{n,2})'| \leqslant d^{n-1}\left(||D_2''||_\infty\left(\sum_{j=1}^{n}\sum_{k=0}^{n-j} d^k\right)||D_1-D_2||_\infty + n||D_1'-D_2'||_\infty\right),$$

or

$$\left|\left(\tau_n^1-\tau_n^2\right)'\right| \leqslant d^{n-1}(n/(1-d))||D_2''||_\infty||D_1-D_2||_\infty + nd^{n-1}||D_1'-D_2'||_\infty. \tag{6.43}$$

From lemma 6.14, equations (6.31), (6.40), (6.41), (6.43), and the convergence of the series $\sum_{n=1}^{\infty} d^{n-1}$ and $\sum_{n=0}^{\infty} nd^n$, the functions D_i', D_i'', D_i''' with $i = 1, 2$ are uniformly bounded and the functions $|G_n|$, $|F_n|$, $n = 0, 1, \ldots$ are uniformly bounded below and above by positive constants. Moreover, there exist constants t_1, t_2, t_3, s_1, s_2 independent of c_e, c_o such that

$$\begin{aligned}|Q_2(\tau)-Q_1(\tau)| &\leqslant (t_1||D_1''-D_2''||_\infty + t_2||D_1'-D_2'||_\infty + t_3||D_1-D_2||_\infty)\\ &\quad\times\sum_{n=1}^{\infty} d^n + s_1||D_1'-D_2'||_\infty + s_2||D_1-D_2||_\infty.\end{aligned} \tag{6.44}$$

Thus, there exist constants p_1, p_2, p_3 independent of c_e, c_o such that

$$|Q_2(\tau)-Q_1(\tau)| \leqslant p_1||D_1''-D_2''||_\infty + p_2||D_1'-D_2'||_\infty + p_3||D_1-D_2||_\infty. \tag{6.45}$$

From lemma 6.8, we have that (6.38) holds . □

Finally, using the previous lemmas we can prove theorem 6.7. Using (4.9), we obtain

$$|g_1'(\tau) - g_2'(\tau)| \leqslant |g_1'(0) - g_2'(0))| + |Q_1(\tau)g_1(\tau) - Q_2(\tau)g_2(\tau)|. \tag{6.46}$$

Now, using (3.14) and the mean value theorem we obtain

$$|g_1'(0) - g_2'(0))| \leqslant |\lambda_1(0)-\lambda_2(0)| < (1/\xi_e)||c_{e1}-c_{e2}||_\infty + (1/\xi_o)|c_{o1}-c_{o2}|,$$

where $\xi_e \in (c_{e1}, c_{e2})$ and $\xi_o \in (c_{o1}, c_{o2})$. Thus, since $c_e \in \mathcal{M}$ and we take c_o satisfying (6.2) there exist constants n_1, n_2 independent of c_e, c_o such that

$$|g_1'(0) - g_2'(0))| = |\lambda_1(0)-\lambda_2(0)| < n_1||c_{e1}-c_{e2}||_\infty + n_2|c_{o1}-c_{o2}|$$

Thus,

$$|g_1'(\tau) - g_2'(\tau)| \leqslant n_1||c_{e1}-c_{e2}||_\infty + n_2|c_{o1}-c_{o2}| + |Q_1(\tau)g_1(\tau) - Q_2(\tau)g_2(\tau)|. \tag{6.47}$$

Finally, using (3.16) and (6.38), we obtain inequality (6.5).

7. Numerical experiments

In [2] and [18] a robust, simple and fast recovery method for the filtration function based on (3.12) was presented, with the limitation that non-monotone filtration functions were obtained unless the experimental data are preprocessed. Examples in this section show that carefully made approximations of the experimental data are necessary to obtain physically acceptable output.

We propose to start the recovery process by choosing an analytic function that not only approximates the experimental effluent concentration, but also satisfies the sufficient conditions for theorems 4.5, 6.7 and assumptions 4.1, 4.2, 6.1, 6.2. Trying to satisfy these sufficient (but

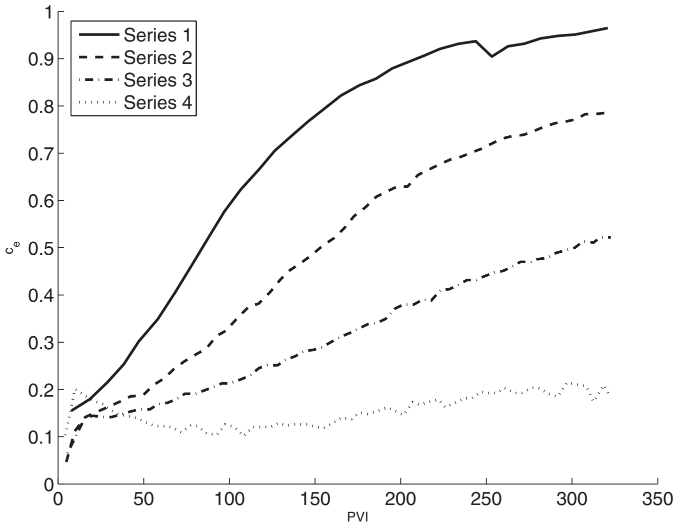
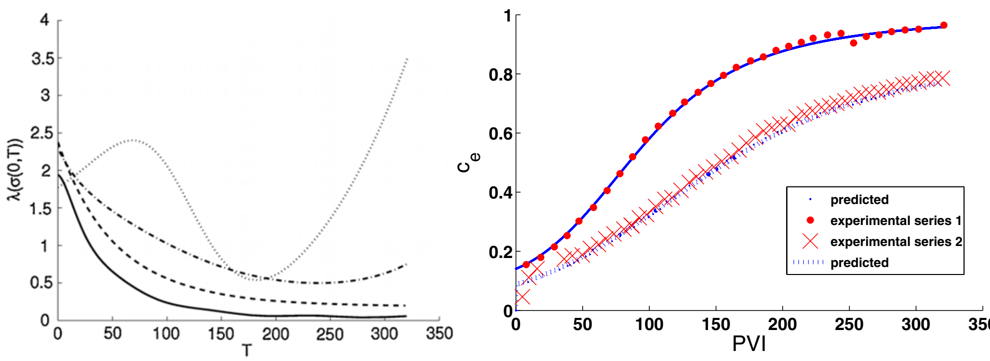


Figure 1. Picture shows the effluent experimental data taken from [15].



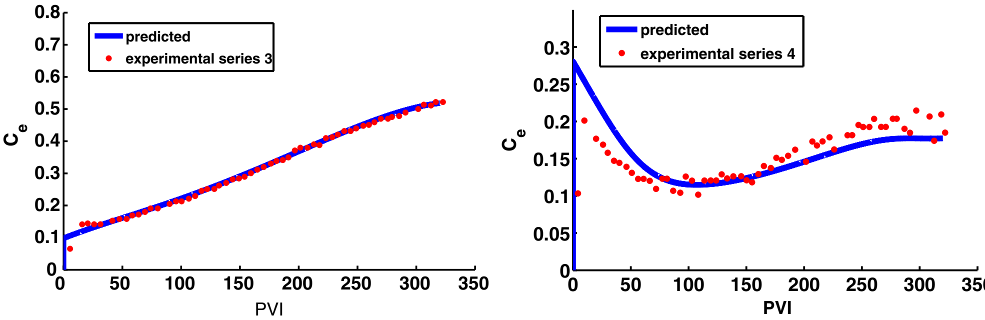
(a) Filtration functions $\lambda(\sigma)$ for the four series.

(b) Recovered effluent profiles for series 1, 2.

Figure 2. The filtration functions shown on the left figure were obtained by solving the inverse problem with the analytic approximations for the data series shown in figure 1; these filtration functions were used to produce figures 2(b) and 3(a)–(b), which show profiles obtained by solving the direct problem. Note the good agreement in figures 2(b) and 3(a)–(b).

not necessary) conditions aims at reducing the number of attempts in preprocessing, which require human intervention and is the weakest part of our procedure. To demonstrate this procedure we choose the same experimental data used in [2], presented in figure 1.

The first step is to fit several analytic curves to the experimental data so that assumption 6.2 is satisfied and then submit them to a test to see if the assumptions 4.1, 4.2 and 6.1 hold. In our experiments, it was verified that the Weibull curve $(a - b\exp(-cx^d))$ approximated well the first series in figure 1 with parameters $(a = 0.95, b = 0.8, c = 2.9 \times 10^{-4}, d = 1.71)$ and the second series with parameters $(a = 0.83, b = 0.73, c = 1.6 \times 10^{-4}, d = 1.68)$. The third series is approximated by $(a + bx + cx^2 + dx^3 + ex^4)$ with $(a = 0.099, b = 0.0012, c = -2.83 \times 10^{-6}, d = 2.61 \times 10^{-8}, e = -5.3 \times 10^{-11})$. The fourth one is approximated by $(a + b\cos(cx + d))$ with $(a = 0.157, b = 0.041, c = 0.175, d = 1.26)$. It is possible to verify that such adjusted curves satisfy the above-mentioned assumptions. In the experiments, we



(a) Recovered profiles for series 3.
(b) Recovered profiles for series 4.

Figure 3. The predicted and experimental series of effluent concentrations. Note the good agreement.

verified that the limits (3.12) converge rapidly. This is to be contrasted to the case when spline approximations for the data were used in [2], in which the convergence was slow.

In the first three experimental series, the data were approximated by $c_e(T)$ such that its derivative is positive (see figure 1), then we checked that the function g given by (3.12) is monotone increasing, leading to monotone decreasing filtration functions λ given in (3.16) (see series 1, 2, 3 in figure 2(a)). On the other hand, in the fourth series (see figure 3), we see that since $c_e(T)$ is non-monotone we obtain the peculiar non-monotone profile for λ shown in figure 2(a) as a dotted curve. The data in the fourth series, as is, does not conform to our model for filtration theory.

8. Conclusions

In this paper, we provided sufficient conditions that guarantee monotonicity of the filtration function obtained by a recovery procedure based on an effluent concentration history obtained by proper preprocessing of the data. Stability conditions based on intrinsic properties of the functional equation were found. We also presented a good numerical method for solving the direct filtration problem, for large times. These are issues that were left pending in [2], now properly solved. The reader interested in short times should use the model and direct method presented in [2], modifying the inversion procedure to include the preprocessing presented here.

The analyticity and monotonicity properties established here for the recovered filtration function justify the optimization method proposed in [17], which imposed parameter-dependent analytical expressions for the filtration function.

We also provided a numerical method to solve the direct problem, which is particularly efficient so it can be used in combination with an optimization procedure as in [2], when monotonicity and analyticity of the effluent concentration and of the filtration function are not valid.

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